Vol. 84, No. 5, December 2011

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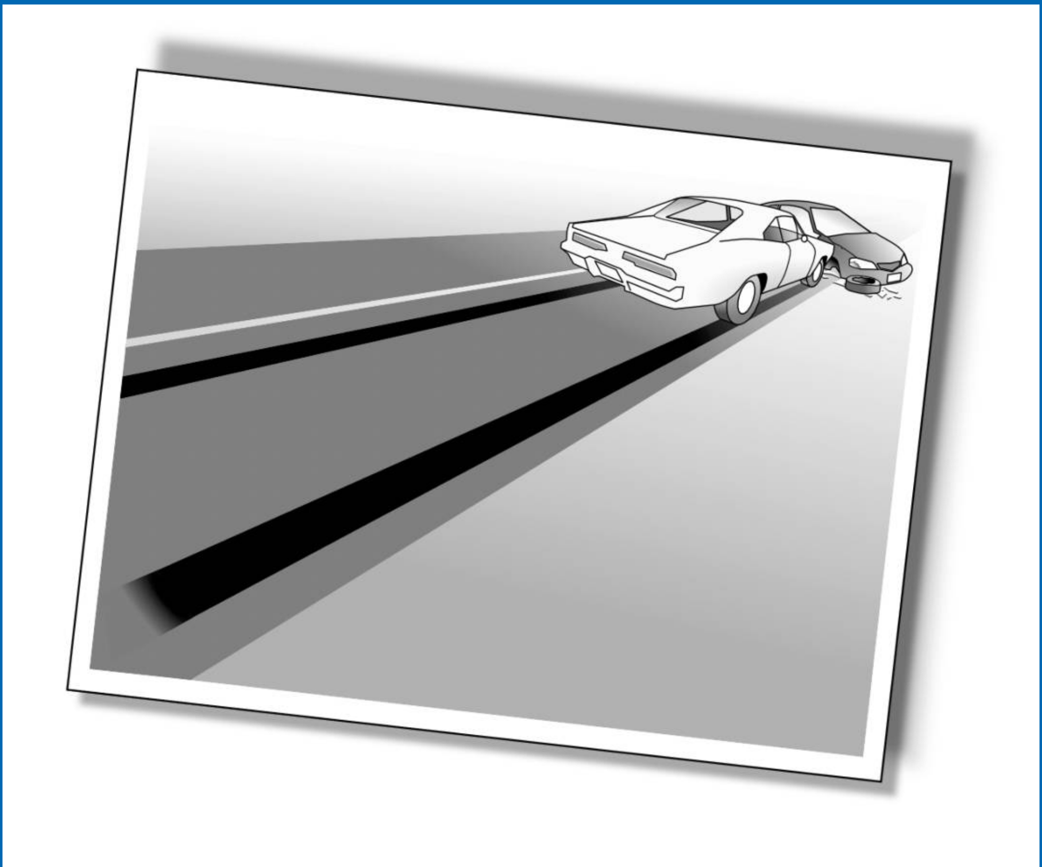
MATHEMATICS MAGAZINE



How fast was this car going?

- Eves's Theorem and Skid Mark Analysis
- Polygonal Numbers and Centered Polygonal Numbers
- Hilbert in Missouri

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Cover image by Marc Frantz

MATHEMATICS MAGAZINE (ISSN 0025-570X) is published by the Mathematical Association of America at 1529 Eighteenth Street, N.W., Washington, D.C. 20036 and Lancaster, PA, bimonthly except July/August.
The annual subscription price for *MATHEMATICS MAGAZINE* to an individual member of the Association is $131. Student and unemployed members receive a 66% dues discount; emeritus members receive a 50% discount; and new members receive a 20% dues discount for the first two years of membership.)

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Postmaster: Send address changes to Membership/Subscriptions Department, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036-1385.

Printed in the United States of America

Vol. 84, No. 5, December 2011

MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

We begin this issue with some remarkable applications of Eves's Theorem. In the hands of author Marc Frantz, this simple theorem of projective geometry seems to do everything. It is a key, not only to the skid-mark application shown on our cover, but also to perspective drawing, proving classical theorems of geometry, and visualizing geometric means.

It's an example of how an abstract technique comes to life when we apply it to a problem that has already grabbed our attention. Steven Schlicker gives us another example. His problem involves some classical integer sequences, the polygonal numbers and the centered polygonal numbers. Which numbers appear in both sequences? His technique is Pell's equation. So often, when we know exactly what we want from a technique, it is easier to see how it works.

We are pleased to present David Zitarelli's history of the "golden age" of the University of Missouri mathematics department, when Earle Raymond Hedrick was chair of the department and president of the MAA. (Looking forward, shouldn't we call it the *first* golden age?) It is a study of Hilbert's influence on the American mathematical community. By telling the story in rigorous detail, the author also gives us a fascinating window on the lives and work of our predecessors.

In the Notes we see patterns in numerical differentiation formulas, extreme values of order statistics, and an application of the inclusion-exclusion principle. Ovidiu Furdui proves some new summation formulas for harmonic numbers, connecting them to the zeta function value $\zeta(3)$. We don't usually publish research papers—we have nothing against research, but the goals of a good research paper are often in tension with the goals of good exposition. And in fact, you may find Furdui's note demanding if you are a generalist, or especially if you are a student. But a solid base of calculus will suffice, and you will be rewarded with a level of understanding that only detailed study provides.

Again in this issue we recognize our referees, whose volunteer efforts make the Magazine possible. Much as the editors value their advice, the strongest praise for their work seems always to come from our authors. The annual index is also a reminder of the many authors who have contributed directly to the Magazine. We thank them, along with those whose work didn't find space in this volume; their willingness to share their knowledge is what makes our community what it is. Finally, we especially appreciate all those whose talents and hard work complete the composition, printing, and distribution of the Magazine.

Walter Stromquist, Editor

ARTICLES

A Car Crash Solved—with a Swiss Army Knife

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Look at the accident photo in FIGURE 1. How fast was the white car going? The question has more than academic interest to the author, who once had the experience of being "T-boned" in a car crash. The focus of this article is on the key to unlocking this mystery—a little known gem called Eves's theorem, which is a kind of Swiss Army knife of projective geometry. We'll not only use it to find the speed of the car, we'll use it to revisit classic theorems, illustrate the concept of the geometric mean, and look at windows and other everyday objects in new ways.

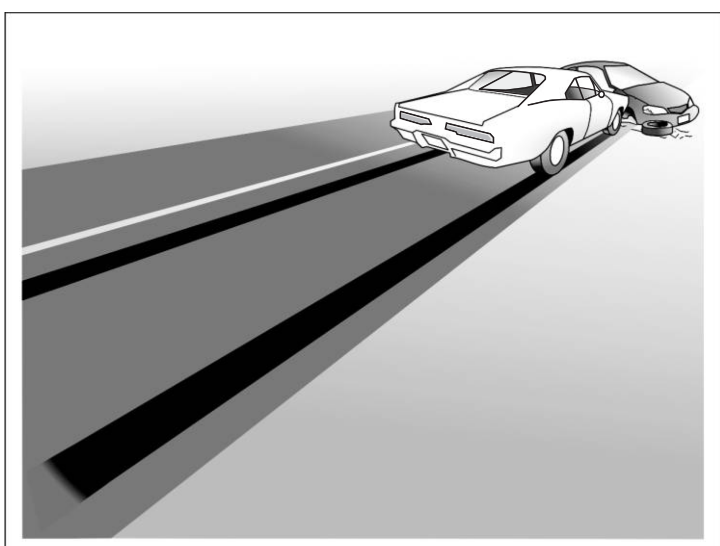

Figure 1 Snapshot of an accident scene. How fast was the white car going?

But first, we refine the car crash question by providing a story to go with the picture, and a little basic physics.

Speed from skid marks

The story goes as follows. The white car and the gray car were headed toward each other in opposite lanes, when the gray car made a left turn in front of the white car

Math. Mag. **84** (2011) 327–338. doi:10.4169/math.mag.84.5.327. © Mathematical Association of America

to enter a parking lot. Upon seeing the gray car making the turn, the driver of the white car slammed on its brakes, locking up its wheels, while managing to hold the car straight in a skid. Unfortunately, the driver of the white car was not able to come to a full stop before striking the gray car in the side as shown. Shortly afterward, a witness snapped the photo in FIGURE 1, just far back enough to show an entire skid mark. The road was repaved a few days later, leaving the photo as the only evidence of the skid marks.

This became important when a dispute arose between the drivers. The driver of the gray car claimed that the white car, a 1969 Dodge Charger, was exceeding the posted speed limit of 35 mi/hr, a claim which the driver of the Charger denied. In addition to the evidence of the photograph, an accident investigator inspected the damaged cars and estimated the speed of the white car at 25 mi/hr at the moment of impact.

There is good news for the driver of the white car: we will give a reasonable analysis that puts an upper bound of 33 mi/hr on the speed of the car at the moment the skid began. We will describe one method of determining the speed—although methods vary in practice—but our greatest emphasis is on showing how Eves's theorem can be used to determine the length of the skid, which is of prime importance in any such analysis.

A little Web searching shows that there are many engineering firms that specialize in accident reconstruction, including skid mark analysis (at least one company provides a "skid speed calculator" [**5**]). We begin by reviewing the problem-solving principles most commonly used when the length of at least one skid mark is known; then we use a rather uncommon method to determine the length of a skid mark in the photograph.

The top part of FIGURE 2 shows a side view of the white car, a 1969 Dodge Charger, along with its skid marks and the specification of its wheelbase (axle-to-axle distance) of 117 inches, or 9.75 feet. The bottom part of FIGURE 2 shows a bird's-eye view of the skid marks, along with a dashed triangle $\triangle ACE$, whose purpose we explain later. The skid mark of the right front tire of the white car ends at point C, but its starting point is obscured by the skid mark of the right rear tire. The skid mark of the right rear tire begins at point A and ends at point B. The distance $|BC|$ is therefore equal to the wheelbase of 9.75 feet. The skid mark of the right rear tire has length $|AB|$. This is the only unobscured skid mark in the witness's photograph.

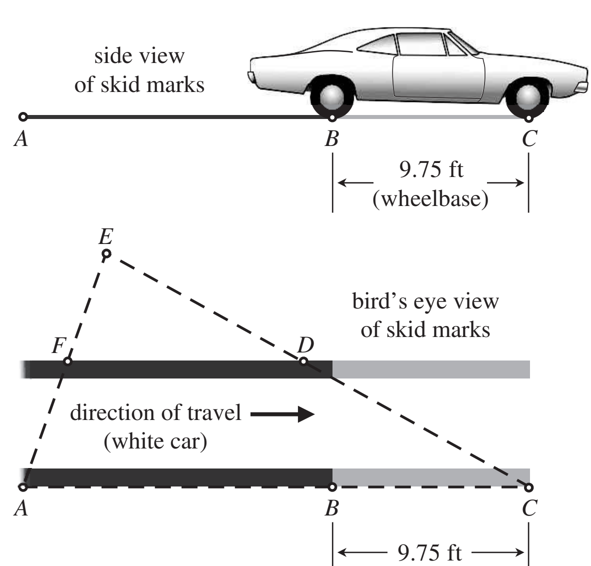

Figure 2 A side view and a bird's-eye view of the skid marks.

Although we don't know the lengths of the other skid marks, it is reasonable to assume they all have length $|AB|$. Let the car have mass m, let v_A denote the car's speed when the right rear tire was at point A, and let v_B denote the car's speed when the right rear tire was at point B. From our earlier information we have an estimate of the impact speed, $v_B \approx 25$ mi/hr, but for the time being we will work with length and time units of feet and seconds. We assume that the road is level, and that during the skid the only external horizontal force acting on the car is the constant deceleration force μmg, where $\mu \geq 0$ is the dimensionless coefficient of sliding friction between tires and road, and $g (\approx 32.174 \text{ ft/s}^2)$ is the acceleration of gravity.

We take the common approach of idealizing the car as a point mass m in rectilinear motion with constant acceleration (see [**4**, pp. 101–102] for example). We assume readers are familiar with two equations from that theory, namely,

$$v - v_0 = at \quad \text{and} \quad x - x_0 = v_0 t + \frac{1}{2}at^2,$$

where x and v are the position and velocity at time t of a particle moving on the x-axis with constant acceleration a, and x_0 and v_0 are the position and velocity at time $t = 0$. By eliminating t between these two equations, we get

$$v_0^2 = v^2 - 2a(x - x_0). \tag{1}$$

This is also a basic equation in the theory of rectilinear motion (see [**4**, Eq. (3-16)] or [**6**, Eq. (3-17)].)

To express (1) in terms of our variables, let the x-axis coincide with the line AB in FIGURE 2, with the origin fixed anywhere, and the positive direction to the right. Denote the x-coordinates of A and B by x_A and x_B, respectively. We model the car as a point mass m that moves from x_A at time $t = 0$ to x_B at time t, under a constant acceleration $-\mu g$. Referring to (1), let $x_0 = x_A$, $x = x_B$, $v_0 = v_A$, $v = v_B$, and $a = -\mu g$. Equation (1) then becomes

$$v_A^2 = v_B^2 + 2\mu g(x_B - x_A),$$

or equivalently,

$$v_A^2 = v_B^2 + 2\mu g|AB|. \tag{2}$$

For computational convenience, it is common to express equation (2) in a hybrid form, with v_A and v_B expressed as respective miles-per-hour speeds $\hat{v}_A$ and $\hat{v}_B$, and $|AB|$ expressed in feet. The conversion factor is $k = (3600 \text{ s/hr})/(5280 \text{ ft/mi})$, so we multiply equation (2) by k^2 to obtain

$$\hat{v}_A^2 = \hat{v}_B^2 + 2k^2\mu g|AB|. \tag{3}$$

At this point we need a value for $2k^2\mu g$. Since we are interested in an upper-limit value of $\hat{v}_A$, we use $\mu = 1$, a widely accepted upper bound for this application. We compute

$$2k^2\mu g \leq 2\left(\frac{3600}{5280}\right)^2 (1)(32.174) \approx 29.91,$$

which we round up to 30, again in the interest of obtaining an upper limit. (Readers will find that this constant 30, whose units are $\text{mi}^2\text{ft}^{-1}\text{hr}^{-2}$, appears in many of the basic skid mark analyses on the Internet.) Substituting 30 for $2k^2\mu g$ in (3) and then taking the square root of both sides, we obtain

$$\hat{v}_A < \sqrt{\hat{v}_B^2 + 30|AB|}.$$

Using the investigator's estimate of $\hat{v}_B \approx 25$ mi/hr, this becomes

$$\hat{v}_A < \sqrt{625 + 30|AB|}, \tag{4}$$

where again, $\hat{v}_A$ is in units of miles per hour, and $|AB|$ is in feet. The inequality (4) gives an upper bound $\hat{v}_A$ on the speed of the white car when the skid began, based on the length $|AB|$ of the skid mark visible in the photo.

Everything depends on the length of that skid mark.

Skid marks from Eves's theorem

In the next section we discuss Eves's theorem in detail. In this section we emphasize how easy it makes finding the skid mark length $|AB|$, and hence the speed of the car.

In FIGURE 2 we have drawn a dashed triangle $\triangle ACE$, where A and C are as described earlier, and E is an arbitrary point on the far side of the skid marks from AC. Side AC contains the point B mentioned earlier, and D and F are the points where the respective sides CE and EA meet the outside edge of the car's left skid mark. Since the skid marks are parallel, we have $|CD|/|DE| = |FA|/|EF|$, hence

$$\frac{|AB|}{|BC|}\frac{|CD|}{|DE|}\frac{|EF|}{|FA|} = \frac{|AB|}{9.75 \text{ ft}} \cdot 1 = \frac{|AB|}{9.75 \text{ ft}}. \tag{5}$$

The expression on the left hand side of (5) is an example of a *circular product* [**7**]. Given a closed polygon with a point in the interior of each side, one forms a circular product by alternately dividing and multiplying consecutive segment lengths, proceeding clockwise or anticlockwise around the polygon. As discussed in the next section, Eves's theorem implies that circular products are *projectively invariant*, which means that if the points A, B, C ... are mapped projectively (as in say, a photograph) to distinct, respective points A', B', C', ..., then the corresponding circular product will have the same value. Specifically, we locate in FIGURE 3 the corresponding points A', B', C', ... as they would appear in the photograph. We need not worry about locating E' perfectly, because it must correspond to *some* preimage point E as in FIGURE 2.

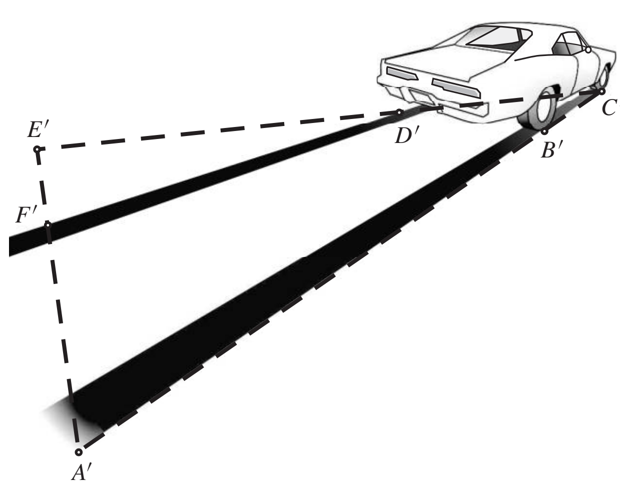

Figure 3 The corresponding image of the triangle in FIGURE 2, as it would appear in the accident photo. For clarity we show the entire side $C'E'$, rather than having part of it disappear under the car.

Having done this, we now use a ruler to measure directly on the photograph in FIGURE 3, and estimate the numerical value of the corresponding circular product. The author did this on a larger image and computed value of just under 1.5 (readers' results will of course vary). Using (5) and the invariance of circular products, we then have

$$\frac{|AB|}{9.75 \text{ ft}} = \frac{|AB|}{|BC|}\frac{|CD|}{|DE|}\frac{|EF|}{|FA|} = \frac{|A'B'|}{|B'C'|}\frac{|C'D'|}{|D'E'|}\frac{|E'F'|}{|F'A'|} < 1.5.$$

Hence, rounding up again, we obtain

$$|AB| < (9.75 \text{ ft})(1.5) < 15 \text{ ft}.$$

Finally, we substitute this result into (4) and round up once more to estimate the white car's speed $\hat{v}_A$ at the beginning of the skid:

$$\hat{v}_A < \sqrt{625 + 30(15)} < 33 \text{ mi/hr}.$$

We therefore conclude that the white car was probably *not* exceeding the posted speed limit of 35 mi/hr when the skid began.

If, on the other hand, we were to work on behalf of the driver of the gray car, we would estimate a *lower* bound for v_A, hoping that it would be significantly greater than 35 mi/hr. This is a bit trickier; for example, we would want to estimate a lower bound on the coefficient of friction. Choosing a value of 0 would be unconvincing, so we would need to be more realistic in this case. A realistic estimate of the coefficient of friction depends on the condition of the road surface, the weather, the brand of tires, the condition of the tires, and other factors. Because our emphasis was on the determination of skid mark lengths from a photograph—a key factor in either case—we chose the simpler problem of estimating an upper bound on v_A.

The science of determining 3-D information from the 2-D images in photographs is called *photogrammetry*. Many firms that do accident reconstruction use specially made photogrammetry software to determine skid mark lengths from photographs. In the preceding problem, Eves's theorem allowed us to determine the skid mark length, and thus the car's speed, by simply drawing a triangle on the image of the skid marks and measuring between certain marker points. Unlike other methods for solving such problems (for example, see [**3**]), we did not need to determine the horizon line, vanishing points, or the viewpoint of the photograph.

Eves's theorem—a Swiss Army knife

Having discussed one application of Eves's theorem, it is time to back up a bit and properly discuss the theorem itself. It is a rare occurrence when an important theorem in a field of mathematics goes largely unnoticed (even by many experts in the field) and more amazingly, makes its debut in the pages of a geometry textbook for students with a background in high school mathematics. Nevertheless, that is the case with Eves's theorem. Howard W. Eves (1911–2004) was for many years a professor at the University of Maine and editor of the Elementary Problems section of the *American Mathematical Monthly*. In section 6.1 of his book *A Survey of Geometry* [**1**], Eves presented this simple but powerful theorem, which can be understood and appreciated by anyone. As the distinguished geometer G. C. Shephard wrote [**7**, p. 1280],

> *We feel that Eves' theorem has never been given the recognition it deserves and should be regarded as one of the fundamental results of projective geometry.*

Before stating the theorem, we discuss some terminology. Given two planes π, π' in space and a point O not on either of them, the map that assigns to each point $P \in \pi$ the point $P' \in \pi'$ such that P, P', and O are collinear is called a *perspectivity with center* O. (When OP is parallel to π' it is customary to map P to a *point at infinity*, but we will not need to worry about this in our examples.) The point P' is called the *perspective image* of P. Similarly we can talk about the perspective image in π' of an entire set in π. The two special cases we have in mind are shown in FIGURE 4.

In FIGURE 4(a) the gray arrows in planes π and π' are related by a perspectivity with center O that lies on the opposite side of π' from the arrow in π. We think of a light ray emanating from each point P of the arrow in π, traveling to a viewer's eye at O in a straight line, and on its way piercing the plane π' at the corresponding point P', like passing through a window and leaving an appropriately colored dot on the glass. The arrow on plane π' is the perspective image of that on π. This is the model for perspective drawing and painting developed in the Renaissance. The idea is that if the arrow on π suddenly disappeared, the viewer at O would be unaware of it, because the light rays from the colored dots on π' would still be coming from the same directions as before. Hence (theoretically at least) the painted image on π' is perfectly realistic, as long as the viewer's eye stays at the viewpoint O.

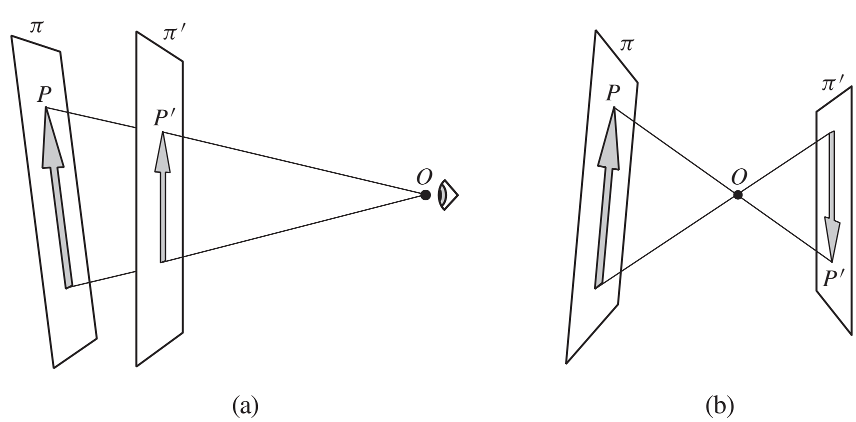

Figure 4 Perspectivities model perspective drawing (a) and photography (b).

FIGURE 4(b) is a simplified model of the photographic process. Here O lies between each pair of corresponding points P, P', causing the perspective image on π' to be inverted. In this model, we think of O as the hole of a pinhole camera, and the line PP' as a light ray passing through it. The arrow on π is an object in the real world, and the inverted image on π' (the screen of the pinhole camera) is its photographic image. Although the structure and function of a lens camera is more complex, to a good approximation the end result is the same—an upside-down perspective image of the given object. Thus for our purposes, we can model both perspective drawing and photography of plane figures with appropriate perspectivities. A composition of perspectivities—for example, a photograph of a photograph—is called a *projectivity*.

We should note that in either case in FIGURE 4, if the planes π and π' are parallel, then the figures in the two planes—the object and its image—are similar. That is, the drawing or photograph is an undistorted likeness of the object, except possibly for resizing. This is the case of the bird's eye view of the skid marks in FIGURE 2.

More generally, however, perspective images are not similar to the real world objects they portray, and it may take some work to recover geometric information about the original object. Eves's theorem shows that there are certain numerical regularities in geometric objects that are *not* changed by the photographic process, such as the circular products associated with the triangles in FIGURES 2 and 3.

There is one other term to explain before stating Eves's theorem. Eves's theorem deals with expressions like the circular product on the left hand side of equation (5), except that each distance such as $|AB|$ is considered to be a *directed distance*, meaning that a positive direction is arbitrarily assigned to the line AB, so that $|AB|$ and $|BA|$ have the same magnitude but opposite signs. This arbitrariness disappears in the expressions under consideration, because every directed distance is divided by, or is divided into, one that is collinear with it. We can therefore think of dealing with signed ratios of collinear pairs of directed distances, a ratio being positive if the two distances are parallel and negative if they are antiparallel. If all collinear pairs have the same direction, then we can simply work with ordinary distances. In [**1**] Eves himself does not explain the origin of the term "h-expression" which follows (but one might think of using it as a mnemonic: "h" for "Howard").

DEFINITION. A product of ratios of directed distances, where all the indicated points lie in one plane, is called an h-expression if it has the following properties:

(1) In each ratio the points that occur are collinear.

(2) Each point appears in the numerator of the product exactly as many times as it does in the denominator.

EVES'S THEOREM. *The value of an h-expression is invariant under any projectivity.*

Observe that the circular product in (5) is an h-expression, which justifies its treatment as a projective invariant in the car problem. Similarly, the well-known cross ratio of four points on a line is an h-expression, hence its projective invariance is a special case of Eves's theorem. Shephard made this observation in [**7**], along with some original applications of the theorem.

This is a significant fact—that the projective invariance of the cross ratio is just a special case of Eves's theorem. We therefore briefly review the cross ratio, and give a simple application of it. The *cross ratio* (AB, CD) of four collinear points A, B, C, D is given by

$$(AB, CD) = \frac{|AC|}{|CB|}\frac{|BD|}{|DA|}, \tag{6}$$

where each quantity such as $|AC|$ is a directed distance. The reader can easily check that (6) is an h-expression. It's important to note that the value of the cross ratio depends not just on the location of the four points, but also on the order of the labels A, B, C, D. Although there are $4!(=24)$ ways to apply the labels to four given points, it's well known that the cross ratio runs through either 3 or 6 different values as the labels are permuted, depending on the points the labels are being applied to.

For practice, we apply the cross ratio to the perspective drawing of a fence in FIGURE 5. Suppose that in some units of length, the distances between the images of the tops of the fenceposts are as indicated in the figure: $|A'B'| = 10$ and $|B'C'| = 6$. Writing $x = |C'D'|$ as in the figure, we ask: what is the value of x?

To answer the question, we let the positive direction be to the right along the fence rail in FIGURE 5, and we use (6) to compute

$$(A'B', C'D') = \frac{|A'C'|}{|C'B'|}\frac{|B'D'|}{|D'A'|} = \frac{16}{-6}\cdot\frac{x+6}{-(x+16)} = \frac{8x+48}{3x+48}. \tag{7}$$

Next we focus on the side view of the fence in FIGURE 6. We label the tops of the fence posts A, B, C, D, and consider the respective points A', B', C', D' in FIGURE 5

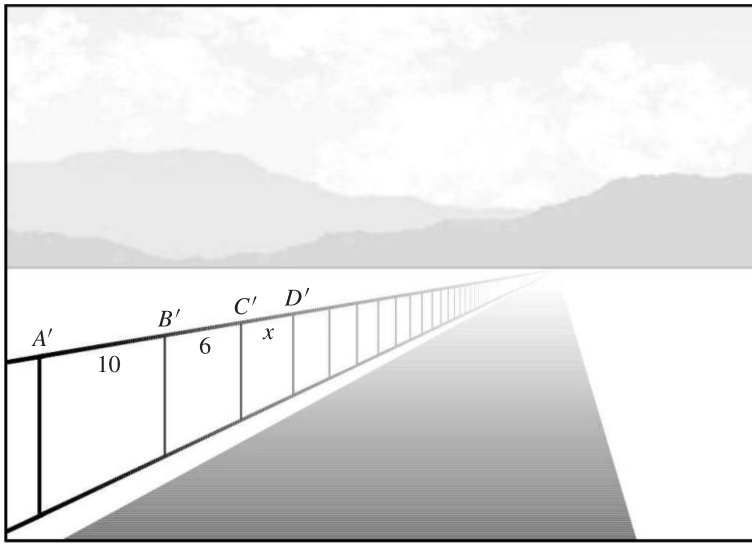

Figure 5 If $|A'B'| = 10$ and $|B'C'| = 6$, then what is $|C'D'|$?

to be their perspective images. Assuming that the fence posts are equally spaced, we indicate in FIGURE 6 that $|AB| = |BC| = |CD| = w$ for some positive number w. Again letting the positive direction be to the right, we have by (6),

$$(AB, CD) = \frac{|AC|}{|CB|}\frac{|BD|}{|DA|} = \frac{2w}{-w} \cdot \frac{2w}{-3w} = \frac{4}{3}. \tag{8}$$

The projective invariance of the cross ratio implies that the results of (7) and (8) are equal, hence

$$\frac{8x+48}{3x+48} = \frac{4}{3}.$$

Solving this for x, we get $x = 4$. (Observe that if only the first three fence posts were drawn in FIGURE 5, we could locate and draw the rest of them by recursively applying this method to the last two known distances between their tops.)

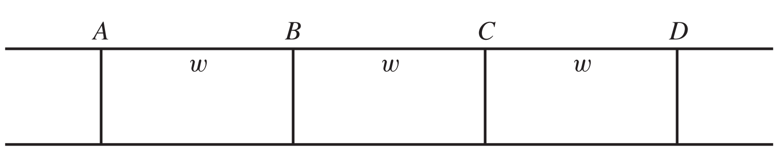

Figure 6 Side view showing equal spacing of the fence posts.

Of course Professor Eves was not unaware of the utility of his theorem. For example, he showed [**1**, p. 290] how to apply the theorem to the proof of the classic theorems of Ceva and Menelaus. Ceva's theorem says that if points D, E, F lie on the respective sides BC, CA, AB of a triangle $\triangle ABC$ as in FIGURE 7(a), the lines AD, BE, CF are concurrent at a point G if and only if the corresponding circular product around the triangle satisfies

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = 1.$$

Menelaus' theorem says that the previously mentioned points D, E, F are collinear as in FIGURE 7(b) if and only if

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = -1.$$

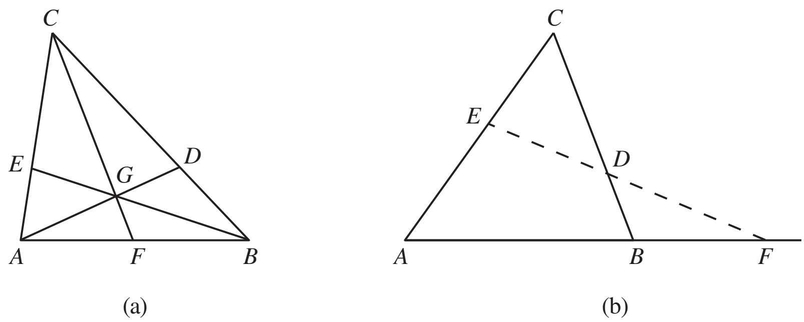

Figure 7 Diagrams for the theorems of Ceva (a) and Menelaus (b).

To give an example of how Eves's theorem applies, Eves proved [**1**, p. 288] that given a configuration like that in FIGURE 7(a), there exists a perspectivity that maps the points $A, B, \ldots$ to respective points $A', B', \ldots,$ such that G' is the centroid of $\triangle A'B'C'$. Since the lines $A'D'$, $B'E'$, $C'F'$ are concurrent at G', the points D', E', F' are midpoints of their respective sides. The "only if" part of Ceva's theorem then follows from an application of Eves's theorem; given the concurrency at G in FIGURE 7(a), we have

$$\frac{|AF|}{|FB|}\frac{|BD|}{|DC|}\frac{|CE|}{|EA|} = \frac{|A'F'|}{|F'B'|}\frac{|B'D'|}{|D'C'|}\frac{|C'E'|}{|E'A'|} = (1)(1)(1) = 1.$$

For projective geometry enthusiasts, we give the following hint for using Eves's theorem to prove the "only if" part of Menelaus' theorem. Suppose the points D, E, F are collinear in FIGURE 7(b). Then what line can be projectively mapped to infinity to give the result

$$\frac{|A'F'|}{|F'B'|} = \frac{|B'D'|}{|D'C'|} = \frac{|C'E'|}{|E'A'|} = -1?$$

The geometric mean in perspective

It is well known that the geometric mean $\mathrm{GM}(x_1, \ldots, x_n)$ of n nonnegative numbers $x_1, \ldots, x_n$ is defined as

$$\mathrm{GM}(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

Unfortunately for visual thinkers, most of the common pedagogical examples of the geometric mean don't admit a visual interpretation for values of n greater than 2 or 3. Eves's theorem makes it easy to construct an example of the geometric mean that includes a picture for each value of n greater than 2. FIGURE 8(a) shows a perspective image of one regular pentagon inscribed in another, with a little thickness added to suggest say, a decorative tile. Suppose we want to use this picture to construct the undistorted bird's-eye view in FIGURE 8(b), in which the vertices of the smaller pentagon divide each side of the larger one into lengths p and q. Given the perspective view in (a), we could make a scale drawing of the bird's-eye view in FIGURE 8(b) if we could just determine the value of the *edge ratio* p/q. By Eves's theorem, the circular product $\prod_{i=1}^{5}(p_i/q_i)$ associated with FIGURE 8(a) must be equal to the circular product $(p/q)^5$ associated with FIGURE 8(b), hence the edge ratio p/q satisfies

$$\frac{p}{q} = \mathrm{GM}\left(\frac{p_1}{q_1}, \ldots, \frac{p_5}{q_5}\right).$$

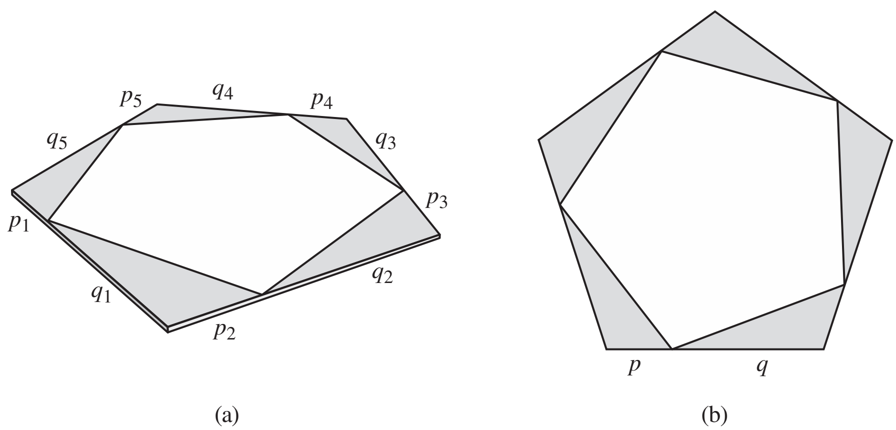

Figure 8 A perspective view (a) and an undistorted bird's-eye view (b) of one regular pentagon inscribed in another. The edge ratio p/q is the geometric mean of the five corresponding edge ratios in the perspective view.

More generally, suppose that for $n \geq 3$ we have a regular n-gon inscribed in a larger one, so that each vertex of the smaller n-gon divides a side of the larger one into lengths p and q. Then, given a perspective drawing or photograph of the configuration in which the image of the ith side of the outer n-gon is divided into corresponding nonzero lengths p_i, q_i for $i = 1, \ldots, n$, the "true" edge ratio p/q is the geometric mean of those in the perspective image:

$$\frac{p}{q} = \mathrm{GM}\left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right). \tag{9}$$

Experimenting with Eves's theorem

A nice feature of Eves's theorem is that it is evident in many of the photographs we see every day in magazines, newspapers, and on the Internet. That's because it is possible to associate h-expressions with many everyday objects such as tile floors, brick walls, windows, parking lot markings, athletic fields, and much more. The exact values of the h-expressions are easy to deduce from the designs of the objects. The goal of such an experiment is to see if the predicted value does indeed result when we make careful measurements on a given photograph. All we need is a photograph of such an object from an interesting angle, a ruler marked in fine graduations such as millimeters, and a little mathematical curiosity.

As a simple example, FIGURE 9 shows a photograph of a large window at a retail store. The inset of the figure is a qualitative front view of the window; the segments *HD* and *BF*, which represent the dividers between the windowpanes, are parallel to the sides *AC* and *CE*, respectively. It should be easy for the reader to verify that

$$\frac{|AB|}{|BC|}\frac{|CD|}{|DE|}\frac{|EF|}{|FG|}\frac{|GH|}{|HA|} = 1,$$

since $|AB| = |FG|$, and so on. In the photograph, the perspective images of these points are labeled with primed versions of the same letters, and according to Eves's theorem, we must have

$$\frac{|A'B'|}{|B'C'|}\frac{|C'D'|}{|D'E'|}\frac{|E'F'|}{|F'G'|}\frac{|G'H'|}{|H'A'|} = \frac{|AB|}{|BC|}\frac{|CD|}{|DE|}\frac{|EF|}{|FG|}\frac{|GH|}{|HA|} = 1.$$

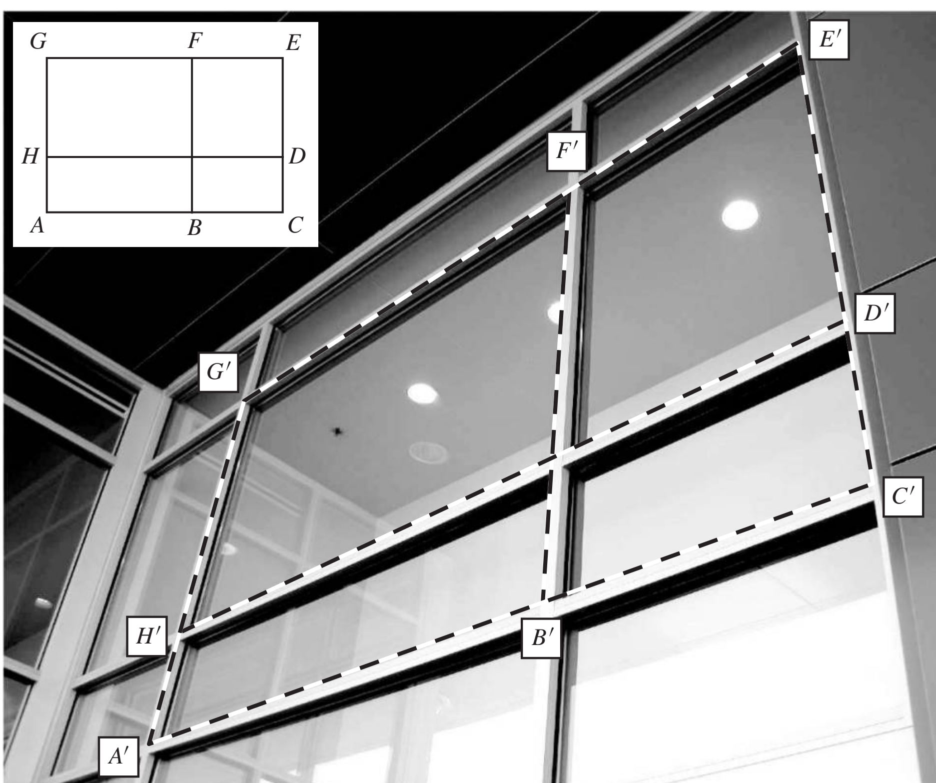

Figure 9 Photograph of a window, and a front view (inset).

We invite the reader to carefully measure the indicated lengths along the dashed lines with, say a ruler marked in millimeters, and then perform the above computation. (Since we must deal with mathematical lines, we drew the dashed lines to represent certain edges of the window frame and the windowpane dividers.) If such measurements are done carefully enough, the result should be reasonably close to 1. Is it?

Exercise Look around you right now. How many objects do you see that you could apply Eves's theorem to?

Conclusion

It is interesting to note that among the pictorial examples of the geometric mean appearing in the literature are some nice ones presented in Professor Eves's last published paper, which appeared in this MAGAZINE [**2**]. We hope the applications presented here, made possible by his theorem, are ones he would have enjoyed.

Acknowledgment The author is grateful to Annalisa Crannell and Fumiko Futumura for several helpful discussions.

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Summary Though not well known, Eves's theorem is a fundamental result of projective geometry—a tool as versatile as a Swiss Army knife. Named for the late Howard W. Eves (1911–2004), the theorem establishes a class of numerical projective invariants, of which the famous cross ratio is a special case. We illustrate the versatility of Eves's theorem by applying it to accident scene reconstruction, to the circular products in the theorems of Ceva and Menelaus, and to perspective illustrations of the geometric mean. In addition, we show that the theorem is illustrated by everyday photographs of buildings and other common objects.

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Numbers Simultaneously Polygonal and Centered Polygonal

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Polygonal numbers are figurate numbers that are generated by constructing larger and larger regular polygons from a fixed vertex. They are called figurate numbers because they can be represented by regular geometrical arrangements of uniformly spaced points. The triangular numbers are examples of polygonal numbers and are probably familiar to most readers. The centered polygonal numbers are a less well known family of figurate numbers, this time generated by arranging points into a sequence of nested polygons of increasing size with a common center. FIGURE 1 illustrates the triangular, square, pentagonal, and hexagonal numbers and their centered counterparts. We let $k \geq 3$ denote the number of sides of a polygon and $p_k(r)$ and $c_k(q)$ the rth k-polygonal and qth k-centered polygonal number, respectively, for $r, q \geq 1$. FIGURE 1 also indicates how figurate numbers are generated by counting the marked points on the nested polygons.

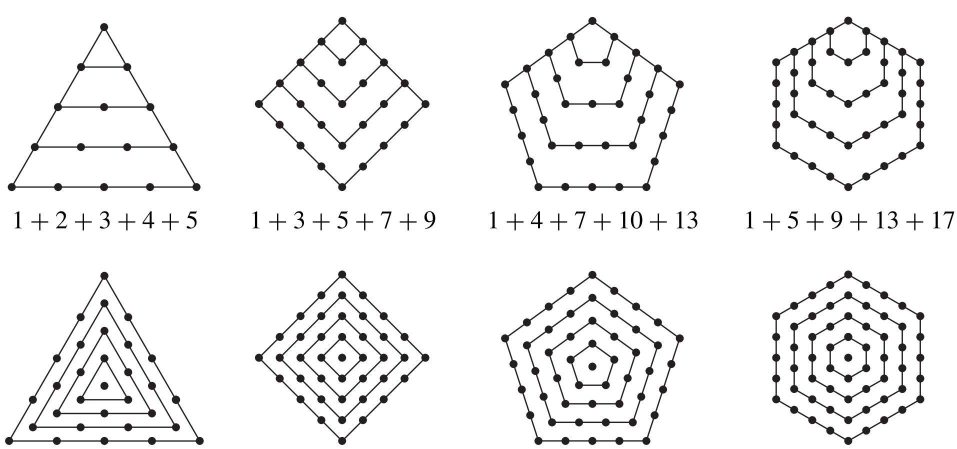

Figure 1 Top: triangular, square, pentagonal, and hexagonal numbers. Bottom: centered triangular, square, pentagonal, and hexagonal numbers.

The On-Line Encyclopedia of Integer Sequences (OEIS) [**17**] lists k-gonal and k-centered polygonal number sequences for k from 3 to 24. The first few entries for some of these sequences are given in TABLE 1. There is a great deal of interesting and fun mathematics, both old and new, connected to these sequences. For example, if we

Math. Mag. **84** (2011) 339–350. doi:10.4169/math.mag.84.5.339.

TABLE 1: Polygonal and centered polygonal number sequences

	polygonal numbers	centered polygonal numbers
triangular	1, 3, 6, 10, 15, 21, 28	1, 4, 10, 19, 31, 46, 64
square	1, 4, 9, 16, 25, 36, 49	1, 5, 13, 25, 41, 61, 85
pentagonal	1, 5, 12, 22, 35, 51, 70	1, 6, 16, 31, 51, 76, 106
hexagonal	1, 6, 15, 28, 45, 66, 91	1, 7, 19, 37, 61, 91, 127
heptagonal	1, 7, 18, 34, 55, 81, 112	1, 8, 22, 43, 71, 106, 148

juxtapose two triangles as illustrated in FIGURE 2 (with $r = 5$), then we obtain an r by $r + 1$ rectangular array of nodes. This implies that

$$p_3(r) = 1 + 2 + 3 + \cdots + (r - 1) + r = \frac{r(r+1)}{2}. \tag{1}$$

As another example, the sum of the marked points in the squares in FIGURE 1 gives us the familiar identity

$$p_4(r) = 1 + 3 + 5 + \cdots + (2r - 1) = r^2.$$

This paper will tell the story of some new mathematics related to these figurate numbers. We will see how the author found a family of new sequences that relates the polygonal numbers to the centered polygonal numbers. Our journey will involve geometry, Pellian equations, and linear algebra, and conclude by identifying the numbers that are both polygonal and centered polygonal for the same number of sides.

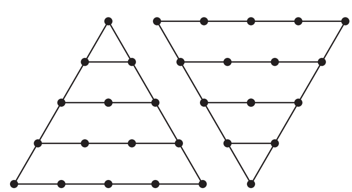

Figure 2 Triangular numbers in two ways.

The polygonal and centered polygonal numbers

Our story begins with the polygonal numbers and centered polygonal numbers. We can derive closed forms for both $p_k(r)$ and $c_k(q)$ by considering how these numbers are constructed. According to Dickson [**5**, p. 1], Hypsicles described $p_k(r)$ as the rth partial sum of the arithmetic sequence beginning at 1 with common difference $k - 2$. So the rth k-polygonal number is the rth partial sum of the series $\sum_{n\geq 1}[1 + (n-1)(k-2)]$. Therefore,

$$\begin{aligned} p_k(r) &= \sum_{n=1}^{r} [1 + (n-1)(k-2)] \\ &= r + (k-2)\left(\frac{(r-1)r}{2}\right) \\ &= \left(\frac{k-2}{2}\right) r^2 - \left(\frac{k-4}{2}\right) r. \end{aligned} \tag{2}$$

In a similar manner, $c_k(q)$ is the qth partial sum of the sequence that begins with 1, k, and then increases arithmetically by k. Thus,

$$\begin{aligned} c_k(q) &= 1 + \sum_{n=1}^{q-1} nk \\ &= 1 + k \sum_{n=1}^{q-1} n \\ &= 1 + k\left(\frac{(q-1)q}{2}\right). \end{aligned} \tag{3}$$

There are many other ways to derive formulas for these figurate numbers. FIGURE 1 shows that there is a recurrence relation that defines the rth polygonal number in terms of the $(r-1)$st polygonal number. Notice that at each successive stage we add $k-2$ sides with r nodes each, but double count the $k-3$ corners. So for $r \geq 1$ we have $p_k(r) = p_k(r-1) + r(k-2) - (k-3)$. A similar recurrence formula exists for the centered polygonal numbers. In this case, we add k sides with q nodes each, but then have to subtract the k corners that are double counted. Thus, for $q \geq 1$ we have $c_k(q) = c_k(q-1) + (q-1)k$. We can then use (1) to derive formulas for $p_k(r)$ and $c_k(q)$ from these recurrence relations. Another method is indicated in Figure 3 which implies that

$$p_k(r) = p_3(r) + (k-3)p_3(r-1)$$

and

$$c_k(q) = p_3(q) + (k-2)p_3(q-1) + p_3(q-2).$$

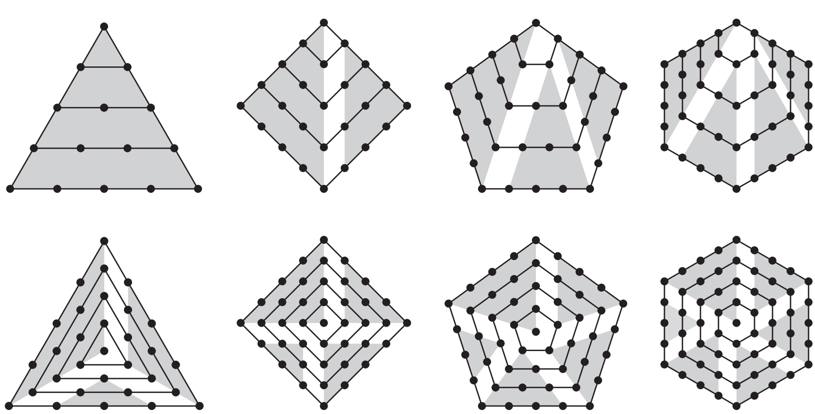

Figure 3 Top: Polygonal numbers as sums of triangular numbers. Bottom: Centered polygonal numbers as sums of triangular numbers.

Aside from being interesting as mathematical objects, polygonal and centered polygonal numbers have recently found applications to cluster science in chemistry, where certain two and three dimensional polygonal numbers appear to provide the most compact and symmetrical arrangements for atoms to cluster [**20**].

Many famous mathematicians have investigated figurate numbers. Dickson [**5**, Chapter 1] gives an extensive account of some of the work done in this area. Polygonal numbers date back to the Pythagoreans and were also studied by Roman and Hindu mathematicians. Dickson cites contributions in the area by Fermat, Euler, Cauchy, Pascal, Legendre, and Leibniz, to name just a few. As an example, Fermat famously asserted [**8**, I, p. 305] that every integer is triangular or the sum of 2 or 3 triangular numbers; square or the sum of 2, 3, or 4 square numbers; pentagonal or the sum of 2, 3, 4, or 5 pentagonal numbers, etc. Cauchy later proved this claim [**3**], and a short proof can be found in [**14**]. Fermat's assertion is probably the most important result concerning polygonal numbers and he promised a book on this subject which, unfortunately, never materialized [**5**, p. 6], [**10**, p. 188]. As related by Polya in the book [**16**, Chapter VI], pentagonal numbers also play a critical role in Euler's beautiful pentagonal number theorem that connects an infinite product to an infinite sum:

$$(1-x)(1-x^2)(1-x^3)\cdots$$
$$= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} + \cdots .$$

The pentagonal numbers occur as every other exponent on the right side of this equation. The sequence of all the exponents is the sequence of *generalized pentagonal numbers* (sequence A001318 in the OEIS).

The question of which numbers can be polygonal in more than one way has a long history. For example, Diophantus purports to have solved the problem of finding in how many different ways a number can be polygonal [**10**, p. 254–259] and Euler investigated polygonal numbers that are also square numbers by reducing the problem to one of making a quadratic a square [**6**]. Gill found numbers that are both m-gonal and n-gonal [**9**, p. 220–225] and Anderson considered the problem of which numbers satisfy $p_m(r) = p_n(q)$ for positive integers m, n, r, and q, giving special treatment to the analysis of numbers for which $p_3(r) = p_n(q)$ and $p_4(r) = p_n(q)$ [**1**].

In this paper we consider the related question of which numbers are simultaneously k-polygonal and k-centered polygonal. As TABLE 1 shows, in addition to $p_k(1) = c_k(1)$ we also have $p_3(4) = 10 = c_3(3)$, $p_4(5) = c_4(4)$, $p_5(6) = c_5(5)$, and $p_6(7) = c_6(6)$. A quick calculation shows that $p_k(k+1) = (k^3 - k^2 + 2)/2 = c_k(k)$, so there are instances besides the case of $p_k(1) = c_k(1) = 1$ when a k-polygonal number is also a k-centered polygonal number.

In short: For which triples k, r, q do we have $p_k(r) = c_k(q)$? That problem will engage us for the rest of the paper.

Setting up the problem

In this section, our story leads to the topic of Pellian equations. To see how, let $k \geq 3$ be a fixed constant representing the number of sides of our polygons. As we saw at the end of the previous section, there are non-trivial instances where numbers are both k-polygonal and centered polygonal. In subsequent sections we will determine exactly which k-polygonal numbers are also k-centered polygonal numbers.

Using formulas (2) and (3) to equate $p_k(r)$ and $c_k(q)$ gives us the equation

$$\left(\frac{k-2}{2}\right) r^2 - \left(\frac{k-4}{2}\right) r = 1 + k\left(\frac{q(q-1)}{2}\right). \tag{4}$$

If we multiply both sides of (4) by $8(k-2)$, add $(k-4)^2$, then complete the squares we obtain the equation

$$(2(k-2)r - (k-4))^2 = 2k + k(k-2)(2q-1)^2. \tag{5}$$

If we let $x = 2(k-2)r - (k-4)$ and $y = 2q - 1$, then (5) becomes

$$x^2 - my^2 = 2k, \tag{6}$$

where $m = k(k-2)$. Equation (6) succinctly describes our problem and holds the key to resolving it. We will need solutions to (6) that also provide integer values of r and q. For q to be an integer we will need to have y be odd. To obtain integer values for r will require having $x + (k-4) \equiv 0 \pmod{2(k-2)}$. So our ultimate task is to find positive integer solutions to (6) with y odd and $x + (k-4) \equiv 0 \pmod{2(k-2)}$. Equations like (6) in which we only allow integer solutions are called *Diophantine equations*.

Equation (6) is also an example of what is called *Pell's equation*, named for the mathematician John Pell. (The attribution was apparently a mistake on Euler's part, who confused the work of Lord Brouncker on this equation in Wallis' *Opera Mathematica* with Pell's contributions to the same work [**7**].) Problems on these types of equations date back to the Greek mathematicians' search for rational approximations to $\sqrt{N}$ for nonsquare N [**21**, Chap. I, VIII]. These equations also appear in the works of Diophantus [**19**] (but for rational solutions), and in the work of the seventh century Indian astronomer and mathematician Brahmagupta to determine when $Cx^2 + 1$ is a square [**4**]. Weil [**21**] provides an interesting history of this equation.

Solving the Diophantine equation $x^2 - my^2 = 2k$

In this section we find all pairs (x, y) of positive integers satisfying equation (6) with odd values of y. In the next section we will show that all of these solutions also have $x + (k-4) \equiv 0 \pmod{2(k-2)}$, which means that they correspond to integer values of r and q. Recall that in our Pellian equation we have $k \geq 3$ and $m = k(k-2)$, and note that m is not a square since $m = (k-1)^2 - 1$.

It may be of interest to review the general theory of Pellian equations, and we will do so shortly. Readers who are familiar with that theory should feel free to skip to the portions of this section that deal with our specific Pellian equation $x^2 - my^2 = 2k$. However, before we begin with the general theory, we make two important observations about the equations $x^2 - my^2 = 1$ and $x^2 - my^2 = 2k$. The first is that the pair $(x, y) = (k, 1)$ is a solution to $x^2 - my^2 = 2k$. The second is that the pair $(x, y) = (k-1, 1)$ is a solution to the related equation $x^2 - my^2 = 1$ (called a *fundamental solution*). As we will see, these special solutions will help us find the general solutions.

We take this opportunity to review the theory of Pellian equations in general. The theory of Pellian equations of the special form $x^2 - dy^2 = 1$ (with d nonsquare) is well known, and its solutions are closely related to those of the general Pellian equation $x^2 - dy^2 = C$ for any integer C. To find the solutions to $x^2 - dy^2 = 1$ with $x, y > 0$ we first find the *fundamental solution*, which is a solution with minimal x (and, as is easily shown, minimal y as well). Let $(x, y) = (a_1, b_1)$ be the fundamental solution. Then all solutions with $x, y > 0$ have the form (a_n, b_n) where a_n and b_n satisfy

$$a_n + b_n\sqrt{d} = \left(a_1 + b_1\sqrt{d}\right)^n$$

for positive integers n. Proofs of this can be found in many number theory texts [**2, 15**].

The fact that the Pellian equation $x^2 - dy^2 = 1$ has a fundamental solution—and hence infinitely many solutions—for each nonsquare integer d appears to have been first shown by Lagrange in [**13**]. Fundamental solutions for the nonsquare integers 2 through 10 are given in TABLE 2. Finding a fundamental solution is a tricky business. Fundamental solutions do not behave in any nice or predictable way. For example, the fundamental solution for $d = 72$ is $(17, 2)$ while the fundamental solution for $d = 73$ is $(2281249, 267000)$. Then for $d = 74$ we have a fundamental solution $(3699, 430)$. In [**11**] the authors show that the size ratio (defined as $\log X_1 / \log X_0$, where X_0 and X_1 are the x values of fundamental solutions corresponding to consecutive values $d - 1$ and d of nonsquare integers) can be arbitrarily large. We are fortunate in the case of $d = m = k(k - 2)$ to have the fundamental solution $(k - 1, 1)$.

TABLE 2: Fundamental solutions of $x^2 - dy^2 = 1$

d	2	3	5	6	7	8	10
fundamental solution	(3, 2)	(2, 1)	(9, 4)	(5, 2)	(8, 3)	(3, 1)	(19, 6)

Solutions to the general equation $x^2 - dy^2 = C$, for an arbitrary positive integer C, are related to the solutions of $x^2 - dy^2 = 1$. We state the relationship in a lemma:

LEMMA 1. *Let d and C be positive integers and (a, b) a solution to $x^2 - dy^2 = 1$. Let (w_0, z_0) be any solution to $x^2 - dy^2 = C$ and define (w_n, z_n) by*

$$w_n + z_n\sqrt{d} = (a + b\sqrt{d})^n (w_0 + z_0\sqrt{d})$$

for each integer n. Then $(x, y) = (w_n, z_n)$ is a solution to $x^2 - dy^2 = C$ for each integer n.

Proof. First notice that the lemma is true for $n = 0$. Now we show that the lemma is true for $n \geq 1$. We have $w_1 + z_1\sqrt{d} = (aw_0 + bz_0 d) + (az_0 + bw_0)\sqrt{d}$ and

$$\begin{aligned} w_1^2 - dz_1^2 &= (aw_0 + bz_0 d)^2 - d(az_0 + bw_0)^2 \\ &= (a^2w_0^2 + 2aw_0bz_0d + d^2b^2z_0^2) - d(a^2z_0^2 + 2az_0bw_0 + b^2w_0^2) \\ &= a^2(w_0^2 - dz_0^2) - db^2(w_0^2 - dz_0^2) \\ &= (a^2 - db^2)(w_0^2 - dz_0^2) \\ &= C. \end{aligned}$$

The step from (w_n, z_n) to (w_{n+1}, z_{n+1}) works in the same way, so (w_{n+1}, z_{n+1}) is a solution to $x^2 - dy^2 = C$ by induction. So our lemma is true for all non-negative integers. The proof for negative integers n can be done in a similar manner and is left to the reader. ∎

A cautionary note: The lemma does not promise to find all solutions to the general equation, nor does it promise that solutions exist. For example, if d is a square and $C = 1$, then $x^2 - dy^2 = x^2 - (\sqrt{d}y)^2$ and a difference of integer squares can never equal 1. So it is possible that there is no solution at all for some pairs d, C. In addition, we will see later that there may be two or more starting solutions that, when used in the lemma, give separate sequences of solutions.

Now we provide the positive integer solutions to (6) that can have odd values of y. We will generate solutions to (6) using the fundamental solution $(k - 1, 1)$ of $x^2 - my^2 = 1$ and the particular solution $(k, 1)$ of (6).

THEOREM 1. *Let $k \geq 3$ be an integer, and let $m = k(k - 2)$. Let $\alpha = (k - 1) + \sqrt{m}$ and $\beta = k + \sqrt{m}$. For each nonnegative integer n, define (x_n, y_n) by*

$$x_n + y_n\sqrt{m} = ((k - 1) + \sqrt{m})^n (k + \sqrt{m}) = \alpha^n \beta. \tag{7}$$

Then (x_n, y_n) *is a solution to* $x^2 - my^2 = 2k$ *with* $x_n, y_n > 0$ *and* y *odd. Moreover, every solution to* $x^2 - my^2 = 2k$ *with* $x, y > 0$ *and* y *odd has the form* (x_n, y_n) *for some nonnegative integer* n.

Proof. Let n be a nonnegative integer. It is clear that $x_n, y_n > 0$. Note that the pair $(x_0, y_0) = (k, 1)$ satisfies (6), so Lemma 1 shows that (x_n, y_n) also satisfies (6).

Since $x_{n+1} = (k-1)x_n + k(k-2)y_n$ and $y_{n+1} = x_n + (k-1)y_n$, an induction argument can be used to show that y_n is always odd and that x_n always has the same parity as k.

Now we need to show that every solution pair (x, y) satisfying (6) with $x, y > 0$ and y odd is of the form (x_n, y_n) for some nonnegative integer n. First notice that if $k = 2t^2$ for some integer $t \geq 2$, then $2k = (2t)^2$ is a perfect square. In this case, the pair $(2t, 0)$ is a solution to (6) and by Lemma 1 generates a second class of solutions to (6)—but we will see that all of the solutions generated in this way have even y values. Note that if $2k$ is not a perfect square, then there are no integer solutions to (6) with $y = 0$.

Now we will show that if (x, y) satisfies (6) with $x > 0$, then $x + y\sqrt{m}$ is an increasing function of y. If $x^2 - my^2 = 2k$, then $x^2 = 2k + my^2$. So if both x and y are positive, then x increases as y increases. Thus, $x + y\sqrt{m}$ is increasing as y increases if both x and y are positive. Assume now that x is positive and y is negative. Then $x + |y|\sqrt{m}$ increases as x and $|y|$ increase together. In this case,

$$x + |y|\sqrt{m} = x - y\sqrt{m} = \frac{x^2 - my^2}{x + y\sqrt{m}} = (2k)(x + y\sqrt{m})^{-1}.$$

So

$$x + y\sqrt{m} = \frac{2k}{x + |y|\sqrt{m}}.$$

As y increases we know $|y|$ decreases and $x + |y|\sqrt{m}$ decreases. It follows that $x + y\sqrt{m}$ increases as y increases.

To prove that solution pair (x, y) satisfying (6) with $x, y > 0$ and y odd is of the form (x_n, y_n) for some nonnegative integer n, we proceed by contradiction and assume (x, y) is a pair of positive integers with y odd satisfying (6) that is not one of the (x_n, y_n) for any nonnegative integer n. Since $x + y\sqrt{m}$ is an increasing function of y, it follows that there exists an integer w such that

$$x_{w-1} + y_{w-1}\sqrt{m} < x + y\sqrt{m} < x_w + y_w\sqrt{m}.$$

Dividing by α^w gives us

$$\begin{aligned}
\alpha^{-w}\left(x_{w-1} + y_{w-1}\sqrt{m}\right) &< \alpha^{-w}(x + y\sqrt{m}) < \alpha^{-w}\left(x_w + y_w\sqrt{m}\right)\\
\alpha^{-w}\left(\alpha^{w-1}\beta\right) &< \alpha^{-w}(x + y\sqrt{m}) < \alpha^{-w}\left(\alpha^w\beta\right)\\
\alpha^{-1}\beta &< \alpha^{-w}(x + y\sqrt{m}) < \alpha^0\beta\\
x_{-1} + y_{-1}\sqrt{m} &< \alpha^{-w}(x + y\sqrt{m}) < x_0 + y_0\sqrt{m}.
\end{aligned}$$

Note that $y_{-1} = -1$ and $y_0 = 1$. Lemma 1 shows that (x_{-w}, y_{-w}) is a solution to (6), but the only time we have solutions to (6) with a y value between y_{-1} and y_0 (that is, $y = 0$) is when $2k$ is square. Next we show that we can eliminate the solutions that arise in this way.

Suppose that (u, v) is a solution to (6) with v even. Then $u^2 = 2k + mv^2$ and so u is even. Consider the solution (u', v') to (6) that is defined by $u' + v'\sqrt{m} = \alpha(u + v\sqrt{m})$. We have

$$\alpha(u + v\sqrt{m}) = [u(k-1) + vm] + [u + v(k-1)]\sqrt{m},$$

and $v' = u + v(k-1)$ is even. So if $\alpha^{-w}(x + y\sqrt{m}) = \gamma$ provides a solution to (6) with $y = 0$, then $x + y\sqrt{m} = \alpha^w\gamma$ has an even value of y, a contradiction. We conclude that (x_n, y_n) provide all the solutions to (6) with $x, y > 0$ and y odd. ■

Finding integer values of r and q

Our solutions (x_n, y_n) to (6) will only provide integer solutions to (5) if, in addition to y_n being odd, we have $x_n + (k-4) \equiv 0 \pmod{2(k-2)}$. We show that this is indeed the case in this section. A little linear algebra will help us out.

Equation (7) can be written in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = M \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \tag{8}$$

where $M = \begin{pmatrix} k-1 & m \\ 1 & k-1 \end{pmatrix}$. The characteristic polynomial of M is

$$x^2 - 2(k-1)x + 1,$$

so we know that $M^2 - 2(k-1)M + 1 = 0$. It follows that

$$\begin{aligned} \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} &= M^2 \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \\ &= (2(k-1)M - 1) \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} \\ &= 2(k-1) \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}. \end{aligned}$$

We are only interested in the x coordinate and

$$x_{n+1} = 2(k-1)x_n - x_{n-1}. \tag{9}$$

From (9) we can obtain

$$\begin{aligned} x_{n+1} + (k-4) &= 2(k-2)x_n + 2x_n + 2(k-4) - x_{n-1} - (k-4) \\ &= 2(k-2)x_n + 2[x_n + (k-4)] - [x_{n-1} + (k-4)]. \end{aligned} \tag{10}$$

Now we proceed by induction. Recall that $x_0 = k$ and $y_0 = 1$. So $x_0 + (k-4) = 2(k-2)$ and we see that $2(k-2)$ divides x_0. Similarly, $x_1 + (k-4) = (2k^2 - 3k) + (k-4) = 2k^2 - 2k - 4 = 2(k-2)(k+1)$. Thus, we have that $2(k-2)$ divides x_1. If $2(k-2)$ divides $x_n + (k-4)$ and $x_{n-1} + (k-4)$, then (10) shows that $2(k-2)$ divides $x_{n+1} + (k-4)$. By induction we conclude that $x_n + (k-4) \equiv 0 \pmod{2(k-2)}$ and each solution (x_n, y_n) to (7) yields integer values of r and q.

We summarize these results in a theorem.

THEOREM 2. *Let $k \geq 3$ be an integer. Then $p_k(r) = c_k(q)$ if and only if r and q are given by*

$$r = \frac{x_n + (k-4)}{2(k-2)} \quad \text{and} \quad q = \frac{y_n + 1}{2}$$

for some nonnegative integer n, where x_n and y_n are defined (as in Theorem 1) by

$$x_n + y_n\sqrt{m} = ((k-1) + \sqrt{m})^n (k + \sqrt{m}).$$

Sequences of k-PC numbers

Our story concludes by constructing sequences of integers that are simultaneously k-polygonal and k-centered polygonal. We will call these sequences k-PC (for Polygonal and Centered polygonal) sequences. Recall that to find integers r and q so that $p_k(r) = c_k(q)$, we let $x = 2(k-2)r - (k-4)$ and $y = 2q - 1$ where (x, y) satisfies $x^2 - k(k-2)y^2 = 2k$. We found that the positive solutions of $x^2 - my^2 = 2k$ (with $m = k(k-2)$) that yield integer values for r and q are of the form (x_n, y_n) given by (7).

Once x_n and y_n are determined, then $p_k(r_n) = c_k(q_n)$ for

$$r_n = \frac{x_n + (k-4)}{2(k-2)} \quad \text{and} \quad q_n = \frac{y_n + 1}{2}. \tag{11}$$

A list of the first few k-PC numbers along with their reference numbers in the OEIS is shown in TABLE 3. We use (7) to find values of x_n and y_n, and (11) to find the corresponding values of r_n and q_n. For example, recall that $x_0 = k$ and $y_0 = 1$. So $x_1 = 2k^2 - 3k$ and $y_1 = 2k - 1$. Therefore, $r_1 = k + 1$ and $q_1 = k$. This reflects the fact we saw earlier that $p_k(k+1) = c_k(k)$, as illustrated in the $n = 1$ row of TABLE 3. We can find the r_n and q_n values for any row by continuing this process. As an example, (7) shows that $x_2 = 4k^3 - 10k^2 + 5k$ and $y_2 = 4k^2 - 6k + 1$ and (11) gives us $r_2 = 2k^2 - k + 1$ and $q_2 = 2k^2 - 3k + 1$. So $p_k(2k^2 - k + 1) = c_k(2k^2 - 3k + 1)$ as illustrated in the $n = 2$ row of TABLE 3.

TABLE 3: Polygonal and centered polygonal numbers

k	3 (A128862)	4 (A008844)	5 (A128917)	6 (A006244)
$n = 0$	1	1	1	1
$n = 1$	10	25	51	91
$n = 2$	136	841	3151	8911
$n = 3$	1891	28561	195301	873181
$n = 4$	26335	970225	12105501	85562821

Rates of growth

As an epilog to our story, we explore the asymptotic behavior of our PC-sequences. This excursion will involve some linear algebra, especially eigenvalues and eigenvectors. Notice that the entries in the k-PC sequence appear to grow very quickly as n increases. Denote the nth entry in the k-PC sequence as $pc_k(n)$. TABLE 4 shows successive quotients of entries from TABLE 3. As this data indicates, the k-PC sequences seem to exhibit roughly exponential growth. To understand this rate of growth, recall that we can write (7) in the matrix form shown in (8). The matrix system (8) is a special case of the general system

$$\begin{pmatrix} w_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_n \\ z_n \end{pmatrix}, \tag{12}$$

which defines a *recurrence relation*—determining the values of w_{n+1} and z_{n+1} in terms of previous values. System of the this type appear in a variety of situations in mathematics: for example in predator-prey models in discrete dynamical systems, Markov processes, modeling duopolies (market environments in which two manufacturers produce identical or interchangeable products) in economics, or modeling feedback in digital signal processing. The system (12) is also called a system of *difference equations*.

TABLE 4: Ratios of successive k-PC sequence entries

k	3	4	5	6	7
$\frac{pc_k(5)}{pc_k(4)}$	13.928081	33.970554	61.983867	97.989795	141.99296
$\frac{pc_k(6)}{pc_k(5)}$	13.928194	33.970563	61.983867	97.989795	141.99296
$\frac{pc_k(7)}{pc_k(6)}$	13.928203	33.970563	61.983867	97.989795	141.99296

We can use techniques from linear algebra (see [**18**, Ch. 5.3] for example) to analyze the system (12). If we begin with some initial condition $(w_0, z_0)^\top$, then a straightforward induction argument shows that $(w_n, z_n)^\top = M^n(w_0, z_0)^\top$, where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Now if the matrix M has two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with corresponding eigenvalues λ_1 and λ_2, then $M^n\mathbf{v}_i = \lambda_i^n\mathbf{v}_i$ for $i = 1, 2$. Because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, we can find scalars α and β so that $(w_0, z_0)^\top = \alpha\mathbf{v}_1^\top + \beta\mathbf{v}_2^\top$. Then

$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = M^n \begin{pmatrix} w_0 \\ z_0 \end{pmatrix} = M^n(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha\lambda_1^n\mathbf{v}_1 + \beta\lambda_2^n\mathbf{v}_2. \tag{13}$$

Equation (13) allows us to derive many things about the system (12). For one, instead of using recursion to calculate the values of the sequences w_n and z_n, (13) provides closed forms for them. Specifically, if $\mathbf{v}_1 = (v_{1,1}, v_{1,2})^\top$ and $\mathbf{v}_2 = (v_{2,1}, v_{2,2})^\top$, then $w_n = \alpha\lambda_1^n v_{1,1} + \beta\lambda_2^n v_{2,1}$ and $z_n = \alpha\lambda_1^n v_{1,2} + \beta\lambda_2^n v_{2,2}$. As a familiar example, if we let $w_n = F_{n+1}$, $z_n = F_n$ (where F_n is the nth Fibonacci number), $a = b = c = 1$, and $d = 0$, then

$$z_n = F_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}},$$

where $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$. This is the well-known Binet formula for the Fibonacci numbers. In this way we can generate all of the general Fibonacci sequences as discussed in [**12**] by letting $z_n = w_{n-1}$, $c = 1$, $d = 0$, and choosing a, b, x_0, and y_0 to have any initial values.

We can also use (13) to determine the long-term behavior of dynamical systems if we know the limiting behavior of λ_1^n and λ_2^n as n increases. In our case, recall that $(x_0, y_0)^\top = (k, 1)^\top$ is the smallest positive solution to $x^2 - my^2 = 2k$. The coefficient matrix $\left(\begin{smallmatrix} k-1 & m \\ 1 & k-1 \end{smallmatrix}\right)$ has two distinct, real eigenvalues $\lambda = k - 1 + \sqrt{m}$ and $\overline{\lambda} = k - 1 - \sqrt{m}$ with corresponding eigenvectors $(\sqrt{m}, 1)^\top$ and $(-\sqrt{m}, 1)^\top$. A little algebra shows that the solution to $(k, 1)^\top = \alpha(\sqrt{m}, 1)^\top + \beta(-\sqrt{m}, 1)^\top$ is $\alpha = \left(\sqrt{m} + k\right) / \left(2\sqrt{m}\right)$ and $\beta = \left(\sqrt{m} - k\right) / \left(2\sqrt{m}\right)$. This gives us

$$x_n = \left(\frac{1}{2}\right)\left((\sqrt{m} + k)\lambda^n - (\sqrt{m} - k)\overline{\lambda}^n\right),$$
$$y_n = \left(\frac{1}{2\sqrt{m}}\right)\left((\sqrt{m} + k)\lambda^n + (\sqrt{m} - k)\overline{\lambda}^n\right). \tag{14}$$

So the asymptotic behaviors of x_n and y_n (and r_n and q_n) depend on the behaviors of λ^n and $\overline{\lambda}^n$ as n goes to infinity. Since $k - 1 + \sqrt{m} > 1$, it is clear that $\lambda^n \to \infty$ as $n \to \infty$. Note also that $\overline{\lambda} = (k - 1) - \sqrt{m} = 1/[(k - 1) + \sqrt{m}] = 1/\lambda$ and therefore $\overline{\lambda}^n$ goes to 0 as n increases to infinity. It follows from (14) for large n we have

$$y_n \sim \lambda^n \left(\frac{\sqrt{m} + k}{2\sqrt{m}}\right)$$

and

$$q_n = \frac{y_n + 1}{2} \sim \left(\frac{1}{2}\right)\left[\left(\frac{\sqrt{m} + k}{2\sqrt{m}}\right)\lambda^n + 1\right]. \tag{15}$$

Let $\hat{q}_n$ denote the expression on the right hand side of (15). A direct computation shows that $c_k(\hat{q}_n) = C_1 + C_2\lambda^{2n}$, where $C_1 = 1 - k/8$ and $C_2 = k(\sqrt{m} + k)^2/(32m)$. So our sequence $\{c_k(q_n)\}$ of k-PC numbers grows asymptotically as $C_2\lambda^{2n}$. Note that $(2 + \sqrt{3})^2 \approx 13.928203$, $(3 + \sqrt{8})^2 \approx 33.970563$, $(4 + \sqrt{15})^2 \approx 61.983867$, $(5 + \sqrt{24})^2 \approx 97.989795$, and $(6 + \sqrt{35})^2 \approx 141.99296$, which agree with our entries in TABLE 4.

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Summary The study of polygonal numbers (triangular, square, etc.) has a long and rich history. Similar, but lesser known, are the centered polygonal numbers which have not been as extensively studied. These sequences of figurate numbers contain a wealth of interesting and fun mathematics. As an example, we make connections between the polygonal and centered polygonal numbers to find a previously unknown family of integer sequences that describe those numbers that are simultaneously polygonal and centered polygonal for the same number of sides.

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Hilbert in Missouri

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No, David Hilbert never visited Missouri. In fact, he never crossed the Atlantic. Yet doctoral students he produced at Göttingen played important roles in the development of mathematics during the first quarter of the twentieth century in what was then the southwestern part of the United States, particularly in that state.

It is well known that Felix Klein exerted a primary influence on the emerging American mathematical research community at the end of the nineteenth century by mentoring students and educating professors in Germany as well as lecturing in the U.S. on two occasions (see [**16**] for details) but less is known about those American-born mathematicians who studied under Hilbert. E. T. Bell observed that in the late 1890s "Hilbert was still on his way to the top and absorbed in his own researches. Moreover, he seems to have been somewhat unapproachable, especially to Americans" [**6**, p. 184]. Yet David Hilbert produced more American doctoral students than Klein, most (13) from the period 1899–1910. According to the late Constance Reid (1918–2010), "The Americans at the University [Göttingen] were sufficient in number and wealth to have their own letterhead: The American Colony of Göttingen" [**17**, p. 48]. This colony even included Hilbert's first female student of any nationality—Anne Lucy (Bosworth) Focke.

This essay introduces three Hilbert colonists who formed the nucleus of the vibrant mathematics department at the University of Missouri (Mizzou) during the first two decades of the twentieth century. Moreover, it examines the high-level program this trio constructed that produced several notable figures without benefit of a doctoral program. Colleagues hired during the critical period 1903–1907 are also introduced because their professional careers illustrate various aspects of academic life in America a century ago—such as the necessity of obtaining a Ph.D. for university positions, the role that Chicago played in satisfying that demand, and nepotism rules. The essay also shows the influence exerted by Felix Klein (through advisors Fine, Maschke, Bôcher, and Osgood) and the Chicago school under E. H. Moore on the next generation of American mathematicians.

Hilbert outpost

The University of Missouri was the prime beneficiary of the Hilbert colony during the first part of the twentieth century, when three of his students—O. D Kellogg, E. R. Hedrick, and W. D. A. Westfall—formed the nucleus of its emergent mathematics department. Kellogg and Hedrick are generally identified with Harvard and UCLA, respectively, where they made their marks, yet both got their start at Mizzou. Mathematicians are often identified that way; G. A. Bliss, for instance, is strongly associated with Chicago even though he too began his professional career at Missouri. Since universities where mathematicians launched their careers are often overlooked, it seems appropriate to examine this institution in this period.

Math. Mag. **84** (2011) 351–364. doi:10.4169/math.mag.84.5.351. © Mathematical Association of America

The 1891–1908 term of university president Richard H. Jesse (1853–1921) is described in glowing terms in the authoritative book by F. F. Stephens on Missouri's history. According to Stephens, "President Jesse's discrimination in the recruiting of new members of the faculty so as to secure men of intellectual competency as well as teaching and administrative ability became so well known that in future years people looked back upon his presidency as the Golden Age of the University" [**20**, p. 355]. Our analysis will isolate the period 1903–1918 as the appropriate Golden Age for the Department of Mathematics.

Figure 1 Richard Henry Jesse, President 1891–1908

TABLE 1 displays the tenure at the University of Missouri for the ten appointments made during 1903–1907. In this section we describe the three Hilbertians—Hedrick, Kellogg, and Westfall.

TABLE 1: University of Missouri tenure, 1903–1907

Name	**Tenure**	**Name**	**Tenure**
Hedrick	1903–1924	Haynes	1905–1951
Ames	1903–1925	Kellogg	1905–1918
Bliss	1904–1905	Walker	1905–1911
Ingold	1904–1935	Westfall	1905–1949
Börger	1905–1907	Dunkel	1907–1916

President Jesse appointed Earle Raymond Hedrick (1876–1943) as chair in 1903. Hedrick had received an A.B. from the University of Michigan in 1896 and had taught high school for a year in Wisconsin before enrolling in the graduate program at Harvard, where he excelled under Maxime Bôcher and William Osgood. As a result, he was awarded a Parker Fellowship for study abroad 1899–1901, when he attended lectures by Felix Klein and David Hilbert. He completed his dissertation on differential equations under Hilbert titled *Über den analytischen Charakter der Lösungen von Differentialgleichungen* and then stayed in Europe a third year on a Harvard scholarship, this time studying with Émile Picard, Édouard Goursat, and Jacques Hadamard, among others, in Paris.

Upon returning to the U.S. Hedrick put in a brief stint at the Sheffield Scientific School at Yale before moving to Missouri in 1903 as professor and head of the mathematics department. Except for serving as director of the mathematical educational corps with the American expeditionary force in France for six months during World War I, he retained these positions until 1924. During this 21-year tenure, Hedrick helped found the Southwestern Section of the AMS in 1906 and was a guiding light in its activities for almost two decades. He was also one of the founders of the Mathematical Association of America (MAA) in 1915 and served as its first president. His contributions to the MAA are commemorated today by the Hedrick Lecture Series because "he had an important part in moulding [MAA] policies, and he has ever since then been a most zealous promoter of [MAA] interests, and one of [the MAA's] most valued counselors" [**3**, p. 224]. Hedrick was also elected president of the AMS for two years, 1938–1939. He translated two classic works, Édouard Goursat's famous *Cours d'Analyse* (with Otto Dunkel) and Felix Klein's *Elementarmathematik vom höheren Standpunkte aus* (with C. A. Noble).

In 1924 Hedrick left Missouri to become professor and head of the department at UCLA, known before 1928 as the Southern Branch of the University of California, Los Angeles. He retained this position for 13 years until being appointed vice president and provost for the University of California system, making him the highest administrative officer at UCLA. He retired from that position in 1942, whereupon he returned to mathematics as a visiting professor at Brown with the aim of inaugurating the *Quarterly of Applied Mathematics.* However, no sooner did he reach Providence than providence intervened and he died in February 1943, two months before the first issue appeared.

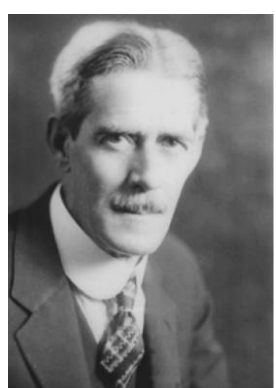

Figure 2 Earle R. Hedrick, Chair of Mathematics, 1903–1924

Bliss and Kellogg are probably the most recognizable names in TABLE 1, but Gilbert Ames Bliss (1876–1951) stayed only one year at Missouri before being recruited by Princeton President Woodrow Wilson for his initial class of preceptors. After spending three years at Princeton 1905–1908, Bliss returned to his *alma mater*, the University of Chicago, where he became one of the world's leading experts on the calculus of variations.

Kellogg's father was an Episcopalian priest who also taught English literature at the University of Kansas. His son Oliver Dimon Kellogg (1878–1932) was inspired to pursue mathematics at Princeton by the Klein protégé, Henry B. Fine. Kellogg earned an A.B. in 1899 and an M.A. the next year before traveling to Germany on a John S. Kennedy Fellowship. After a year in Berlin, he moved to Göttingen, where he received a Ph.D. under David Hilbert in January 1903 for the dissertation "Zur Theorie der Integralgleichungen und des Dirichlet'schen Prinzips." Upon graduation he returned to Princeton for two years before accepting an assistant professorship at Missouri. He thus joined Earle Hedrick as the second member of the Hilbert colony on campus.

Kellogg published several impressive papers on potential theory over the next few years. G. D. Birkhoff stated that for Kellogg "the scientific environment proved happy and stimulating despite a considerable amount of teaching and administrative duties" [**7**, p. 172]. Kellogg was promoted to full professor in 1910, the first time Missouri ever housed more than one full professor of mathematics. In 1909 Kellogg coauthored a textbook with Hedrick, *Applications of the calculus to mechanics.* Yet it was Kellogg's 1912 paper "Harmonic functions and Green's integral" that established his reputation as a first-class researcher. He continued to turn out impressive papers, mainly on orthogonal functions, over the next few years until his research program was interrupted by World War I when he served as scientific advisor to the U.S. Coast Guard Academy in Connecticut from June 1918 to June 1919.

At the end of the war O. D. Kellogg was appointed lecturer at Harvard to succeed Maxime Bôcher, who had recently passed away. Buffeted by joint works with G. D. Birkhoff, especially a paper in 1922 that generalized the Brouwer fixed point theorem, Kellogg rose through the ranks to become full professor in 1927. Two years later his book *Foundations of Potential Theory* became the first by an American in the famous Springer "Yellow Series." He was about to assume the chair at Harvard when he died in August 1932 at age 54 from a heart attack while mountain climbing in Maine.

The other Hilbert student whom President Jesse appointed in 1905 was Wilhelmus David Allen Westfall (1879–1951), a natural attraction for Earle Hedrick for two reasons. First, Westfall and Hedrick had been at Yale together. Westfall received his A.B. in 1901 and Hedrick joined the faculty that fall, with Westfall staying on campus the next two years. The second reason was that Westfall, probably inspired by Hedrick, joined the American colony in Göttingen and completed his doctorate under David Hilbert in 1905 for the dissertation, "Zur Theorie der Integralgleichungen."

Thus Westfall was a fresh Ph.D. when he came to Missouri. During the years 1905–1918 he combined with Hedrick and Kellogg to form *the* Hilbert outpost in the westward expansion of mathematics in the U.S.

Unlike Kellogg and Hedrick, W. D. A. Westfall found the Mizzou surroundings compatible enough to remain on that campus for the rest of his life, retiring in 1949 after 44 years on the faculty. His first few years show how important the German research milieu was for his career. After publishing no papers during his first three years at Missouri, he traveled to Rome in April 1908 with a small group of Americans led by E. H. Moore to attend the International Congress of Mathematicians. Westfall remained in Europe that summer, recharging his research batteries in Göttingen, and producing several papers. Within six months he published one on generalized Fourier coefficients, another on a generalized Green's function, and two short notes extending results of Erhard Schmidt (who had received his Göttingen Ph.D. under Hilbert the same year as Westfall). Moreover, he submitted three others that soon appeared.

In 1906 the three Hilbertians played central roles in establishing the Southwestern Section of the AMS. Hedrick was elected chair; Kellogg and Westfall presented papers. When the Section met officially for the first time the next November, Kellogg was elected secretary, a post he held until his entry into WWI. The Southwestern Section was the third in the AMS, following Chicago in 1897 and San Francisco in 1902; today the four AMS Sections are defined geographically.

A paper that Westfall read at the 1909 sectional meeting was published five years later in the Italian journal *Rendiconti del Circolo Matematico di Palermo.* And in 1916 he coauthored the paper "Sur l'existence des fonctions implicites" with Earle Hedrick in the *Bulletin de la Société Mathématique de France.* Of particular relevance for this account is a joint paper with Hedrick and their colleague Louis Ingold that appeared in the *Journal de mathématiques* in 1923. This collaborative effort was indicative of the close cooperation that had taken place earlier in the Golden Age of the Missouri mathematics department but by the time its major result was extended from two to three dimensions in a 1925 article by Hedrick and Ingold [**13**], Hedrick was at UCLA and such collaboration had run its course. In fact, the 1925 paper was the last of six that Hedrick and Ingold coauthored.

Other appointments

In this section we describe the remaining figures in TABLE 1, beginning with L. D. Ames and Otto Dunkel and then moving to those of less renown, including the above mentioned Louis Ingold. In so doing, we encounter an impressive list of advanced courses, highlight some notable mathematicians who benefited from these offerings, and discuss the delayed implementation of a doctoral program.

As noted, President Jesse brought Hedrick and Ames to campus in 1903. Lewis Darwin Ames (1869–1955) had taken summer courses at the University of Chicago during 1897 and 1898 while teaching at a normal school, but he earned one bachelor's degree from Missouri in 1899 and another from Harvard two years later. He matriculated in the Harvard graduate program for another two years before returning to the University of Missouri in the fall of 1903. Toward the end of his first year in graduate school he read a paper at an AMS meeting in New York that was published in the *Annals* and was cited in the literature as late as 1966 [**4**]. During his second year his improvements to a preliminary version of a paper were duly noted by the appreciative author, E. V. Huntington [**14**, p. 360]. When Ames was awarded his Ph.D. in 1905 he became the first of well-known analyst William Fogg Osgood's four doctoral students.

L. D. Ames read a paper at the AMS annual meeting in his first December back at Mizzou that was published the following March [**1**]. Over the next few years he published papers on the emerging field of *analysis situs* (now topology), with the subject's premier American contributor, Oswald Veblen, referencing his work [**21**, p. 83]. But it seems that Ames soon became involved in educational matters, being elected secretary of the Missouri Association of Teachers in 1906. Four years later his title was changed from assistant professor of mathematics to assistant professor of the teaching of mathematics, which might explain why his output diminished radically after such an auspicious start. Ames left Missouri one year after Hedrick, accepting a position at Texas Tech (then the Texas State College at Lubbock) for 1925–1926. The following year he moved to the University of Southern California, where he remained until his retirement in 1946.

The remaining notable mathematician hired during Jesse's presidency was Otto Dunkel (1869–1951). Due to the necessity of working at a young age, Dunkel did not enter college until age 24, so he was 29 when he left the University of Virginia with bachelor's and master's degrees. He then matriculated at Harvard, earning his Ph.D. in 1902 for a dissertation written under Bôcher. Dunkel then taught at Wesleyan University for two years before spending 1904–1905 at Göttingen (where he met Westfall) and the next year in Paris. Upon returning to the U.S. he taught for a year at the University of Minnesota before accepting the instructorship at Missouri in 1907. He left Mizzou in 1916 for Washington University in St. Louis, where he stayed until retirement in 1939. Dunkel published many papers in several fields but he is best

Figure 3 Otto Dunkel

remembered for serving as editor of the Problems Department of the *Monthly* from 1918 through 1946.

The remaining four mathematicians in TABLE 1 are minor figures in the history of mathematics, yet their careers illustrate critical differences between academic life in America today and a century ago. The first, Louis Ingold (1872–1935), earned his Mizzou A.B. in 1901. Because of rapidly increasing numbers of undergraduate students at the turn of the century, many universities found it expedient to appoint student assistants to perform lower-level teaching. Ingold was one such "teaching assistant" during his senior year. He remained on campus for 1901–1902 as a "teaching fellow," which paid him a small stipend while he earned an A.M. for a thesis titled "Geometry of four dimensions." He enrolled in the graduate program at the University of Chicago the next year but returned to Missouri as an assistant for 1903–1905.

Ingold played an important role at the summer 1904 AMS meeting held in conjunction with the St. Louis World's Fair. Although he presented a paper, his most noteworthy contribution was the construction of physical models similar to the ones Felix Klein had demonstrated at Chicago eleven years earlier. The minutes from the meeting record that "an excursion was made to the palace of education, where Professor Hedrick explained the exhibit of the University of Missouri... Of marked interest was a model made by Mr. Ingold [that illustrated] in red and blue wire a large number of lines representing real and imaginary points of a real circle. Other models related to geodesic lines, subgroups of the modular group, and analysis situs" [**12**, p. 56].

Ingold was granted a leave for 1905–1906 to continue graduate study at Chicago, where he completed his dissertation in 1907 under Klein's former student Heinrich Maschke. Ingold remained on the Missouri faculty the rest of his life, directing five doctoral dissertations and serving as a cooperating editor of the *Transactions of the AMS.*

The fall of 1905 saw the largest increase in mathematics faculty with five fresh bodies. A report by the Board of Curators summarized the situation (as quoted in [**8**, p. 13]):

> In the summer of 1905, G. A. Bliss, assistant professor of mathematics, resigned to accept a position at Princeton. Oliver D. Kellogg has been elected as his successor.... This, together with the large enrollment in mathematics, has made it necessary to appoint two additional instructors in that subject. R. L. Börger and W. D. A. Westfall have been appointed.

We have seen that the Hilbertians Kellogg and Westfall were prominent in university affairs over the next thirteen years.

The other three mathematicians in TABLE 1 had checkered careers at Missouri. Robert Lacey Börger (1873–1932) earned his A.B. at Lake City Agricultural Institute (now the University of Florida) in 1893, taught for a year in Illinois, and then matriculated at Johns Hopkins, where he left without obtaining a degree. He was appointed assistant professor at Lake City in 1896 and promoted to professor two years later but was summarily dismissed in 1904 when the school's new president sought someone with a doctorate and postdoctoral experience. In the meantime Börger had taken graduate courses at Chicago during every summer quarter from 1898 through 1902. He spent the entire year after his dismissal at Chicago, earning a master's degree. That was his background before his appointment as instructor at Missouri in 1905. Paul Ehrlich wrote, "Börger fits the hardscrabble pattern... of working one's way up through the ranks, without the privilege of doctoral study after obtaining the B.A., but continuing graduate work piecemeal during summers, and with an occasional full year of study" [**9**, p. 3]. Two years after coming to Missouri Börger earned a Ph.D. at Chicago under Leonard Dickson for a dissertation on ternary linear groups in a Galois field of order p^n. After obtaining his degree Börger left Missouri for an instructorship at the University of Illinois. He moved to Ohio University in Athens, OH, in 1916 and remained there as head of the department for the rest of his career.

Eli Stuart Haynes (1880–1956) represents the department's older tie to astronomy, being hired as an assistant in mathematics after earning his Missouri A.B. in 1905 but moving to the Laws Observatory the next year. He earned a Ph.D. in astronomy at Berkeley in 1913 and returned to Mizzou ten years later as professor of astronomy and director of the Laws Observatory. His wife is more relevant to this account. Nola (Anderson) Haynes (1897–1996) earned bachelor's and master's degrees at Missouri and then taught in high school and a junior college. She then matriculated in the graduate program at Missouri, which was beginning to mature, and earned her Ph.D. in 1929 under Louis Ingold for the dissertation "An extension of Maschke's symbolism." Recall that Heinrich Maschke was Ingold's thesis advisor at Chicago, making Nola Haynes a third-generation descendant of Felix Klein. After another year as instructor Nola Anderson became chair at H. Sophie Newcomb College (a degree-granting, coordinate college for women at Tulane), a position she held until returning to Missouri in 1938 to marry E. S. Haynes. In a 1981 interview she recalled, "There was a very strict nepotism law and I was giving up my career for marriage, thinking I would never teach again. Then when the Second World War came... I was the first person called back" [**11**, online supplement, Haynes, p. 2]. However, in 1946 she joined the faculty as acting associate professor and when her husband retired five years later, she became the first woman to hold the title of associate professor of mathematics at the university. E. S. Haynes died in 1956; Nola Haynes remained at Missouri until her retirement eleven years later. When she died in December 1996, just 19 days shy of her 100th birthday, her family created the Nola Anderson Haynes Scholarship fund at the university.

Nola Haynes was not the first woman to hold a faculty position at Missouri, nor the first to obtain a doctorate there. The first female faculty member was Mary Shore (Walker) Hull (1882–1952), who had attended Arkansas Industrial College (now the University of Arkansas) before transferring to Missouri in 1900. She earned an A.B. (1903) and A.M. (1904), the latter with the thesis "On finite groups with special reference to Klein's ikosaeder." In 1905 she was appointed assistant in mathematics and promoted to instructor two years later, thus becoming the first woman on Mizzou's mathematics faculty.

Mary Walker's academic record as a student at Missouri illustrates the breadth and depth of departmental offerings at the turn of the twentieth century. She took advanced courses with all three Hilbertians—number theory with Kellogg, real and complex variables with Hedrick, and differential equations with Westfall—as well as Galois Theory (Ames), Lie groups (Bliss), and Fourier series (Defoe). This impressive list of advanced courses ranks among the best offered by American universities at that time.

Mary Shore Walker's later career illustrates the effects of anti-nepotism policies. Because Missouri had not yet awarded a doctorate in mathematics when she earned her master's degree, she obtained leaves to enroll at Yale, where she received her Ph.D. in 1909. While in New Haven she met fellow graduate student Albert Wallace Hull, who obtained his Ph.D. in physics the same year, but she returned to Missouri as an instructor for two years while he taught at Worcester Polytechnic Institute. She was a gifted teacher whose "freshmen classes said she made math sound like poetry" [**2**, p. 20]. However, unlike Nola (Anderson) Haynes, Walker-Hull's academic career ended permanently when she wed in June 1911.

To further assess the success of the Mizzou program a century ago we describe the achievements of some of its outstanding students and the genesis of the doctoral program. Three notable mathematicians who earned bachelor's and master's degrees (like Mary Walker) were subsequently listed among mathematicians awarded stars in *American Men of Science.* Wallie Abraham Hurwitz (1886–1958) earned both degrees in 1906, having made a definite impression on Gilbert Bliss during his junior year: "W. A. Hurwitz was an extraordinarily able and precocious student in one of his classes" [**3**, p. 201]. Upon graduation he enrolled at Harvard 1906–1909 and then traveled to Göttingen for a year, earning a doctorate in 1910 under David Hilbert. But Hurwitz did not join the Hilbert outpost at his alma mater afterwards, instead spending a very productive career at Cornell.

Figure 4 Wallie Hurwitz

Edward Wilson Chittenden (1885–1977) received his Missouri degrees in 1909 and 1910 before earning his Ph.D. at Chicago under E. H. Moore in 1912; he went on to a distinguished career at the University of Iowa. Lester Randolph Ford (1886–1967) received his degrees in 1911 and 1912 before earning a Harvard Ph.D. under Bôcher in 1917 after two years in Paris. His professional career was divided between Rice and the Illinois Institute of Technology. His numerous and longtime MAA activities were recognized when the MAA established the Lester R. Ford Award for authors of expository papers in the *Monthly.* Missouri undergraduates Walker, Hurwitz, Chittenden, and Ford earned Ph.D.s at Yale, Göttingen, Chicago, and Harvard, respectively, thus attesting to the high level of the program constructed by the Hilbert colonists!

Missouri awarded its first Ph.D. in mathematics in 1910 to the Lithuania-born Louis Lazarus Silverman (1884–1967), whose dissertation does not list an advisor. Upon graduation he taught for eight years at Cornell, where he published joint papers with Wallie Hurwitz. Silverman then moved to Dartmouth until retiring in 1953.

The next doctorate was awarded in 1915 to Eula Adeline (Weeks) King (1882–1967), who had taken undergraduate courses at Mizzou while teaching high school, graduating in 1908 as valedictorian of the class. After earning a master's degree the next year, she enrolled in the graduate program at Bryn Mawr College but left after three years without obtaining a degree. Once again she found success at Missouri, earning her Ph.D. for a dissertation supervised by Earle Hedrick, making her a second-generation descendant of Hilbert.

With Ph.D. in hand, the career path that Eula Weeks pursued—high-school teacher—is somewhat surprising. If she felt such a calling, her case would be similar to Anna Mullikin, who chose high-school teaching after earning a doctorate under R. L. Moore and displaying considerable research ability. (See [**5**] for details.) Mullikin remained single but in 1924 Weeks married industrial-arts teacher Harry King, a fellow faculty member at Grover Cleveland H.S. in St. Louis. Up till then she had been very active in the MAA and the embryonic National Council of Teachers of Mathematics, but her professional career ended with her marriage. We have been unable to discover if the dissertations by Louis Silverman or Eula Weeks-King were ever published.

The Missouri Ph.D. program did not mature until the late 1920s, raising the question why the mathematics department did not establish a viable doctoral program in mathematics under Hedrick, Kellogg, and Westfall (plus Dunkel and Ingold). A lack of sufficient finances would seem to be ruled out because Missouri was among sixteen select universities that provided subventions for the financially strapped *Transactions* during its first ten years of existence, 1900–1909. Such a commitment seems to suggest that the University of Missouri sought to position itself as one of the leading mathematics departments in the country.

From 1907 to 1916 the nucleus of the Missouri mathematics faculty included three Hilbert colonists—Hedrick, Kellogg, and Westfall. Also, Dunkel had pursued postgraduate studies at Göttingen. In addition, Hurwitz had also obtained his Ph.D. under Hilbert. When one includes the secondary influence of the Klein sphere mentioned in several places above, one can see the positive influence that Germany exerted on the westward migration of the American mathematical research community.

Missouri up to 1902

In order to place the Golden Age in perspective, we end with a synopsis of Missouri's early history as it relates to mathematics. Although the development of the faculty and course offerings were quite similar to most other land-grant universities from today's Midwest, Mizzou can boast of particularly strong administrative ties to mathematics from the beginning.

The University of Missouri was established in 1839 when the state enacted a fund to originate a state university like those that had been founded in Michigan and Indiana in 1837 and 1838, respectively. John Hiram Lathrop (1799–1866) was appointed the first president, having been strongly recommended by William W. Hudson (d. 1859), the mathematics professor at Columbia College who had been one of Lathrop's classmates at Yale. Up until 1829 Lathrop had been the professor of mathematics and natural philosophy at Hamilton College in New York. Not surprisingly, he chose his friend Hudson to be the first professor of mathematics, with duties extending to natural science (physics) and astronomy as well.

The university's first building was completed in July 1843. Because Lathrop was aware of advances in engineering education that had taken place at West Point, he

Figure 5 William W. Hudson, first Professor of Mathematics

mandated that the sophomore course (in the entirely prescribed curriculum) include applications to leveling and surveying, projections, and navigation.

When Lathrop resigned in 1849 to become the first chancellor at the newly established University of Wisconsin, mathematics professor William Hudson was appointed interim president. Lathrop's successor, James Shannon, made one notable change when he restricted the duties of university tutor Robert A. Grant to mathematics. The rank of tutor was similar to instructor today, and Grant was listed as a faculty member in the college catalog.

But Robert Grant was involved in a much more dramatic incident than being a mere tutor: *shooting a student to death.* In 1856 15-year-old George P. Clarkson was reprimanded for fighting. He placed the blame squarely on Grant, striking him with a cane when they met after the hearing. The faculty reconvened and voted to expel Clarkson at once. Later that day the student and tutor crossed paths on a downtown street, words were exchanged, and pistols were drawn and fired. Clarkson's shot was off the mark but Grant's was accurate; the teenager was critically wounded and died a few days later. At the subsequent public trial Grant was acquitted as having acted in self defense, but Mizzou dismissed him anyway.

Figure 6 Class of 1883 (Note the cane)

Also in 1856, the mathematician William Hudson was appointed president on a regular basis after James Shannon was dismissed because of pro-slavery lectures delivered across the state. Hudson himself was a slaveholder who had accompanied Shannon on many of his lecture tours but he did not proselytize his views publicly. However, poor health soon overcame President Hudson and he died in June 1859. Later the law and medical schools were built on 183 acres from the Hudson tract left from his estate that also included the "Hudson mansion," then the largest private residence in Columbia.

In 1860 Edward F. Fristoe was appointed professor of mathematics and astronomy. This was the beginning of a difficult period in the state of Missouri, which was torn asunder by divided loyalties during the Civil War. Precarious finances forced the university to close the following spring, but the passage of the historic Morrill Act of July 1862 establishing land-grant universities compelled Missouri legislators to reopen the university in the fall to qualify for funding. Conditions became so bad that one senior student, appointed tutor while attending his own classes, became responsible for all instruction in mathematics when Fristoe resigned to fight in the war.

The immediate postwar period was also difficult for the university. One positive development was the establishment in September 1867 of a Normal School devoted to training prospective teachers for public schools. Because most public-school teachers were women, this development had the effect of allowing women to matriculate for the first time; as a result, 22 were admitted in the fall of 1868. Three years later women were allowed to take courses in all curricula.

Mathematics professor Joseph Ficklin (1833–1887) had come to Missouri at the end of the Civil War in 1865. He became the first Missouri mathematician to publish in a professional journal when his short note on calculus appeared in the *Analyst* [**10**]. Ficklin remained at Missouri until his death in 1887, so overall he served as chair for 22 years.

Figure 7 Joseph Ficklin, Professor of Mathematics 1865–1887

In 1876 Samuel Spahr Laws was appointed president. Like initial president Lathrop, Laws had a solid background in mathematics, having been professor of the subject at the newly founded Westminster College, a Presbyterian school located in nearby Fulton. The Laws Observatory was subsequently named in his honor because of a new telescope he purchased with his own funds.

Laws's successor, Richard H. Jesse, had no particular ties to mathematics, yet he was responsible for making Missouri a Hilbert outpost a dozen years after coming to office in 1891. President Jesse's term got off to a rocky start when a fire destroyed the main building in January 1892, just six months after his inauguration, but construction proceeded so quickly that six buildings were completed by the opening of classes in the fall of 1893.

Dramatic increases in enrollment during the 1890s allowed Jesse to make several faculty appointments. Moreover, professors became associated with departments, so

Figure 8 Main Building 1885

Figure 9 Ruins of the Main Building

when the department of astronomy was formed in 1893, mathematics became an entity in its own right. Actually a prized mathematician and an aspiring mathematician of note were already on campus when Jesse came to office in 1891. The aspirant was an unusual student, Cassius Jackson Keyser (1862–1947), who had been a school principal and superintendent before becoming an instructor in mathematics at Missouri in 1889 while studying toward the bachelor's degree he earned in 1892. He became affiliated with Columbia University for the rest of his life, joining the faculty in 1900, receiving his Ph.D. in 1901, and serving as chair of the department 1910–1916. Keyser retired as Adrain Professor Emeritus of Mathematics in 1927.

Figure 10 William Benjamin Smith

Missouri's mathematics professor when Jesse became president was William Benjamin Smith (1850–1934) who, according to the records of the New York Mathematical Society in 1891, earned a Ph.D. in mathematics and physics at Göttingen in 1879. Smith came to Missouri in 1885, and one of his first students was the above mentioned C. J. Keyser, who credited Smith for his success and sought to resurrect Smith's accomplishments 50 years later. (See [**15**, p. 305].) Smith published both an extended review of an important German work [**18**] and a respected textbook [**19**] in 1893, his final year at Missouri. He then left for a higher salary at Tulane, where fifteen years later he switched to the philosophy department and wrote a very controversial book on religion.

Upon Smith's departure, Luther M. Defoe (1860–1933) was appointed temporary head of the mathematics department for the year 1893–1894. An 1893 Harvard graduate, Defoe spent the rest of his life at Missouri except the year 1902–1903 in Cambridge, England. Willoughby Cordell Tindall (1856–1898) was appointed chair in

1894 after Defoe's interim term. He had been an instructor before taking a leave for graduate work at Harvard 1893–1894, but he left without obtaining a degree. Unfortunately, Tindall died at age 42. John N. Fellows succeeded him in 1898; the only thing we know about him is that he departed in favor of Earle Hedrick five years later.

The "Golden Age" that marked President Jesse's tenure at Missouri was hardly evident in mathematics up to 1902. After all, Luther Defoe, Willoughby Tindall, and John Fellows are hardly household names. As we saw, however, Jesse appointments from 1903–1907 confirm his lofty stature. TABLE 1 does not include Arthur Byron Coble (1878–1966), whom Jesse hired as instructor in 1902. Coble had received his Ph.D. at Johns Hopkins under Frank Morley but he returned to Hopkins after just one year at Missouri. In 1918 he accepted a professorship at the University of Illinois, where he remained for the rest of a very productive career in research mathematics. President Jesse brought Earle Hedrick to Missouri to succeed Coble and thereby set the stage for the Hilbert American outpost described in the first part of this article.

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Summary David Hilbert never traveled to the United States yet he exerted considerable influence on the development of mathematics in the country during the first half of the twentieth century through the thirteen Ph.D. students he produced from 1899 through 1910. This article introduces three of those graduates (Earle Hedrick, O. D. Kellogg, and W. D. A. Westfall) who formed the nucleus of the vibrant mathematics department at the University of Missouri 1903–1925 and who played important roles in the expansion of mathematical activities into what was then the southwestern part of the U.S. The impressive curriculum this trio constructed without aid of a Ph.D. program produced several notable mathematicians. Moreover, the careers of some of their colleagues illustrate various aspects of academic life in America a century ago.

DAVID ZITARELLI has been teaching a course at Temple University on the history of mathematics in America for the past 15 years. During one of those classes he discovered that the World's Fair held in St. Louis in 1904 sponsored a Mathematical Congress similar to the more famous Chicago Congress conducted eleven years earlier. He reported on his subsequent investigation in the September 2011 issue of the *Notices of the AMS*, isolating the critical part played by the emerging American Mathematical Society. That article indicated that mathematicians at the University of Missouri played a leading role in the St. Louis Congress, which induced him to study the mathematics department there during the first part of the twentieth century. That research isolated the critical role played by three of David Hilbert's American students, which resulted in the present article.

NOTES

An Upper Bound for the Expected Difference between Order Statistics

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Suppose we randomly and independently choose n numbers $X_1, X_2, \ldots, X_n$ from the interval [0, 1] according to some probability distribution. Put these numbers in ascending order and call the results $Y_1 \leq Y_2 \leq \cdots \leq Y_n$. If $1 \leq k < \ell \leq n$, how large can the number $Y_\ell - Y_k$ be? A moment's reflection reveals that by choosing the X's appropriately, we can make $Y_\ell - Y_k$ as small as zero or as big as one. But what if we consider the *expected value* of the random variable $Y_\ell - Y_k$? This expectation can be as small as zero if the common probability distribution of the X's is degenerate at a single point. But how large can this expectation be? We will answer that question in this article.

In statistics courses the set of random variables $X_1, X_2, \ldots, X_n$ is called a *random sample* and $Y_1, Y_2, \ldots, Y_n$ are called its *order statistics*. We represent the common probability distribution of X_i by the *cumulative distribution function* (cdf), defined for all real numbers x by $F(x) = P(X_i \leq x)$. We assume that $F(0^-) \stackrel{\text{def}}{=} \lim_{x\to 0^-} F(x) = 0$ and $F(1) = 1$ so that $P(0 \leq X_i \leq 1) = 1$.

We first compute the probability distribution of Y_k, as follows. For $x \in [0, 1]$, let N_x be the number of observations among $X_1, X_2, \ldots, X_n$ which do not exceed x. The random variable N_x is therefore the number of successes in n Bernoulli trials, where each trial has success probability $p = F(x)$. Consequently, N_x has a *binomial distribution* and hence

$$\begin{aligned} P(Y_k > x) &= P(N_x \leq k-1) \\ &= \sum_{j=0}^{k-1} \binom{n}{j} [F(x)]^j \, [1 - F(x)]^{n-j} \,. \end{aligned} \tag{1}$$

The expected value of any random variable Y satisfying $P(0 \leq Y \leq 1) = 1$ is given by

$$\mathcal{E}(Y) = \int_0^1 P(Y > y)\, dy \tag{2}$$

(see, for example, Chung [**1**]). Note that this last integral always exists as a real number.

Math. Mag. **84** (2011) 365–369. doi:10.4169/math.mag.84.5.365. © Mathematical Association of America

It now follows from (1) and (2) that

$$\mathcal{E}_F\left(Y_\ell - Y_k\right) = \int_0^1 P_{k,\ell}\left(F(x)\right)\, dx, \tag{3}$$

where our notation indicates that this expectation depends on F and the polynomial $P_{k,\ell}$ is defined for $t \in [0, 1]$ by

$$P_{k,\ell}(t) = \sum_{j=k}^{\ell-1} \binom{n}{j} t^j\, (1-t)^{n-j}\,. \tag{4}$$

The uniform distribution

As an example, suppose that each X_i has the *Uniform distribution* with cdf given by $F(x) = x$ for $0 \le x \le 1$. It follows from (1) that

$$P(Y_k > x) = \sum_{j=0}^{k-1} \binom{n}{j} x^j (1-x)^{n-j}.$$

This distribution is called a *Beta distribution* (see for example, Hogg, McKean, and Craig [**2**]). We leave it as an exercise for the reader to show that

$$\int_0^1 x^j (1-x)^{n-j}\, dx = \frac{1}{(n+1)\binom{n}{j}}. \tag{5}$$

Readers who are familiar with Beta distributions will recognize this result. It now follows from (3), (4), and (5) that

$$\mathcal{E}\left(Y_\ell - Y_k\right) = \frac{\ell - k}{n+1}. \tag{6}$$

The reader should ask herself whether this last result seems intuitively reasonable.

We will next find a distribution for X_i (which will depend on the choice k and ℓ) that maximizes the value of $\mathcal{E}\left(Y_\ell - Y_k\right)$.

A Bernoulli distribution

Suppose each X_i has the probability distribution with point masses at zero (with probability p) and at one (with probability $1 - p$). This distribution is called a *Bernoulli distribution*. Its cdf depends on p and is given by

$$F_p(x) = \begin{cases} 0 & \text{if } x < 0 \\ p & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1. \end{cases}$$

From (3) it then follows that

$$\mathcal{E}_p\left(Y_\ell - Y_k\right) = P_{k,\ell}(p).$$

Which value of p maximizes this expectation? After some algebra it is easy to see that

$$P'_{k,\ell}(p) = np^{k-1}(1-p)^{n-l}\left[\binom{n-1}{k-1}(1-p)^{\ell-k} - \binom{n-1}{\ell-1}p^{\ell-k}\right]. \tag{7}$$

Using the first derivative test it now follows $\mathcal{E}_p(Y_\ell - Y_k)$ is maximized for

$$p = p_{\max} \stackrel{\text{def}}{=} \left(1 + \left(\frac{\binom{n-1}{\ell-1}}{\binom{n-1}{k-1}}\right)^{\frac{1}{\ell-k}}\right)^{-1} \tag{8}$$

and that $\mathcal{E}_p(Y_\ell - Y_k) < \mathcal{E}_{p_{\max}}(Y_\ell - Y_k)$ if $p \neq p_{\max}$. For notational simplicity we have suppressed the dependence of $p_{\max}$ on k and ℓ.

The main result

The last example leads to our main result, which gives the maximum value for $\mathcal{E}_F(Y_\ell - Y_k)$ and answers the question posed in the introduction. We are maximizing this expectation over *all* possible probability distributions on [0, 1].

THEOREM 1. *Suppose $X_1, X_2, \ldots, X_n$ are independent random variables each having the same cdf F satisfying $F(0^-) = 0$ and $F(1) = 1$. For fixed integers k and ℓ satisfying $1 \leq k < \ell \leq n$, let Y_k and Y_ℓ be the kth and ℓth order statistics for $X_1, X_2, \ldots, X_n$. Define the polynomial $P_{k,l}$ as in* (4), $p_{\max}$ *as in* (8) *and $F_{\max}$ by*

$$F_{\max}(x) = \begin{cases} 0 & \text{if } x < 0 \\ p_{\max} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1. \end{cases}$$

Then $\mathcal{E}_F(Y_\ell - Y_k) \leq P_{k,\ell}(p_{\max})$ with equality holding if and only if $F(x) = F_{\max}(x)$ for all x.

Proof. Using (3) and (7) and applying the first derivative test we have

$$P_{k,\ell}(p_{\max}) - \mathcal{E}_F(Y_\ell - Y_k) = \int_0^1 P_{k,\ell}(p_{\max}) - P_{k,\ell}(F(x))\, dx \geq 0. \tag{9}$$

Equality obviously holds if $F = F_{\max}$. Conversely, suppose there is a number $x_0 \in [0, 1)$ at which $F(x_0) \neq p_{\max}$. Since F is right continuous at x_0, it follows that there is a number $\delta > 0$ such that the integrand in (9) is strictly positive on $[x_0, x_0 + \delta)$. Since this integrand is nonnegative on [0, 1], the inequality in (9) must be strict in this case. ■

A possible application and some examples

As a possible application of our theorem, suppose data are collected from some unknown distribution on the interval [0, 1] and the values of Y_k and Y_ℓ are obtained. Since $Y_\ell - Y_k$ is an unbiased estimator of its expectation, an observed value of this difference which grossly exceeds our upper bound may cast doubt on the assumption that our data are a random sample.

The upper bound $P_{k,\ell}(p_{\max})$ simplifies nicely in certain special cases. We explore two cases. The reader is invited to consider others.

CASE 1. $\ell = k+1$

In this case $p_{\max} = \frac{k}{n}$ and so the maximum value for $\mathcal{E}_F(Y_{k+1} - Y_k)$ is

$$P_{k,k+1}(p_{\max}) = \binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}, \tag{10}$$

which is the binomial probability of obtaining exactly k successes in n Bernoulli trials each with success probability $\frac{k}{n}$.

To interpret this maximum value, we point out that according to our theorem, if the upper bound for $\mathcal{E}_F(Y_{k+1} - Y_k)$ is to be achieved, then every one of the X's must be either zero or one. If we interpret successes as zeros and failures as ones, then $Y_{k+1} - Y_k = 1$ when we have exactly k successes and $Y_{k+1} - Y_k = 0$ otherwise. Hence $\mathcal{E}_F(Y_{k+1} - Y_k)$ is the binomial probability of obtaining exactly k successes in n Bernoulli trials. It can easily be checked that the value of the success probability p which maximizes this binomial probability is $p_{\max} = \frac{k}{n}$.

CASE 2. $\ell = n+1-k$ where $k < \frac{n+1}{2}$

In this case $\binom{n-1}{k-1} = \binom{n-1}{\ell-1}$ so that $p_{\max} = \frac{1}{2}$ and hence the maximum value for the expected "trimmed range" $\mathcal{E}_F(Y_{n+1-k} - Y_k)$ is

$$\frac{1}{2^n}\sum_{j=k}^{n-k}\binom{n}{j} = 1 - \frac{1}{2^{n-1}}\sum_{j=0}^{k-1}\binom{n}{j}. \tag{11}$$

Our theorem shows that this maximum is achieved only in the case where

$$P(X_i = 0) = \frac{1}{2} = P(X_i = 1)$$

A different proof of that fact, which uses the notion of convexity, is given in [**3**] for the case $k = 1$.

Returning to the case where each X_i is uniformly distributed and recalling (6) we see from (10) that

$$\frac{1}{n+1} < \binom{n}{k}\left(\frac{k}{n}\right)^k\left(1-\frac{k}{n}\right)^{n-k}$$

and from (11) that

$$\frac{k}{n+1} > \frac{1}{2^n}\sum_{j=0}^{k-1}\binom{n}{j}$$

We close by inviting the reader to verify these last two inequalities directly.

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Summary It is well known that the order statistics of a random sample from the uniform distribution on the interval [0, 1] have Beta distributions. In this paper we consider the order statistics of a random sample of n data points chosen from an arbitrary probability distribution on the interval [0, 1]. For integers k and ℓ with $1 \leq k < \ell \leq n$ we find an attainable upper bound for the expected difference between the order statistics Y_ℓ and Y_k. This upper bound depends on the choice of k and ℓ but does not depend on the distribution from which the data are obtained. We suggest a possible application of this result and we discuss some of its special cases.

Counting Irreducible Polynomials over Finite Fields Using the Inclusion-Exclusion Principle

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Why there are exactly

$$\frac{1}{30}(2^{30} - 2^{15} - 2^{10} - 2^6 + 2^5 + 2^3 + 2^2 - 2)$$

irreducible monic polynomials of degree 30 over the field of two elements? In this note we will show how one can see the answer instantly using just very basic knowledge of finite fields and the well-known inclusion-exclusion principle.

To set the stage, let $\mathbb{F}_q$ denote the finite field of q elements. Then in general, the number of monic irreducible polynomials of degree n over the finite field $\mathbb{F}_q$ is given by Gauss's formula

$$\frac{1}{n}\sum_{d|n} \mu(n/d) q^d,$$

where d runs over the set of all positive divisors of n including 1 and n, and $\mu(r)$ is the Möbius function. (Recall that $\mu(1) = 1$ and $\mu(r)$ evaluated at a product of distinct primes is 1 or -1 according to whether the number of factors is even or odd. For all other natural numbers $\mu(r) = 0$.) This beautiful formula is well-known and was discovered by Gauss [**2**, p. 602–629] in the case when q is a prime.

We present a proof of this formula that uses only elementary facts about finite fields and the inclusion-exclusion principle. Our approach offers the reader a new insight into this formula because our proof gives a precise field theoretic meaning to each summand in the above formula. The classical proof [**3**, p. 84] which uses the Möbius' inversion formula does not offer this insight. Therefore we hope that students and users of finite fields may find our approach helpful. It is surprising that our simple argument is not available in textbooks, although it must be known to some specialists.

Math. Mag. **84** (2011) 369–371. doi:10.4169/math.mag.84.5.369. © Mathematical Association of America

Proof of Gauss's formula

Before we present our proof we collect some basic facts about finite fields that we will need. These facts and their proofs can be found in almost any standard algebra textbook that covers finite fields. See for example [**1**, Chapter 14.3], [**3**, Chapter 7.1, 7.2], or [**4**, Chapter 20.1].

1. A finite field of order q exists if and only if q is a prime power. Moreover, such a field is unique up to isomorphism, and is denoted by $\mathbb{F}_q$.
2. $\mathbb{F}_{q^n}$ is the splitting field of any irreducible polynomial $p(x)$ of degree n over $\mathbb{F}_q$. (This means $p(x)$ factors into linear factors over $\mathbb{F}_{q^n}$ but not over any smaller subfield of $\mathbb{F}_{q^n}$.)
3. The roots of an irreducible polynomial over $\mathbb{F}_q$ are distinct.
4. No two distinct monic irreducible polynomials over $\mathbb{F}_q$ can have a common root.
5. $\mathbb{F}_{q^a} \subseteq \mathbb{F}_{q^b}$ if and only if a divides b.

With these basic facts under our belt we proceed to show that the number of irreducible monic polynomials of degree n over $\mathbb{F}_q$ is equal to

$$\frac{1}{n}\sum_{d|n}\mu(n/d)q^d.$$

The case $n = 1$ is easy because every degree one monic polynomial is irreducible. In fact, the number of degree one monic polynomials over $\mathbb{F}_q$ is equal to q, and this is exactly what we get from the above formula when we plug in $n = 1$. Therefore for the rest of the proof we will assume that $n > 1$. Let $\mathcal{P}_n$ denote the collection of all irreducible monic polynomials of degree n over $\mathbb{F}_q$ and let $\mathcal{R}_n$ be the union of all the roots of all the polynomials in $\mathcal{P}_n$. Note that fact (2) ensures that the roots thus obtained are contained in $\mathbb{F}_{q^n}$. Moreover, using facts (3) and (4) we conclude that $\mathcal{R}_n$ is the disjoint union of n-element sets, one for each polynomial in $\mathcal{P}_n$. Thus,

$$|\mathcal{R}_n| = n\,|\mathcal{P}_n|\,.$$

Therefore, it is enough to compute $|\mathcal{R}_n|$. To this end, observe that

$$\begin{aligned}\mathcal{R}_n &= \{\alpha \text{ in } \mathbb{F}_{q^n} \mid\ [\mathbb{F}_q(\alpha)\colon \mathbb{F}_q] = n\},\\ &= \{\alpha \text{ in } \mathbb{F}_{q^n} \mid \alpha \text{ is not contained in any proper subfield of } \mathbb{F}_{q^n}\},\\ &= \{\alpha \text{ in } \mathbb{F}_{q^n} \mid \alpha \text{ is not contained in any maximal proper subfield of } \mathbb{F}_{q^n}\}\end{aligned}$$

Let $n = u^a v^b w^c \cdots$ be the prime factorization of n with r distinct prime factors (recall that $n > 1$). Then the maximal subfields of $\mathbb{F}_{q^n}$ by fact (4) are of the form

$$F_u = \mathbb{F}_{q^{(n/u)}},\ F_v = \mathbb{F}_{q^{(n/v)}},\ F_w = \mathbb{F}_{q^{(n/w)}},\ \ldots.$$

Then by the third interpretation of $\mathcal{R}_n$ given above, we have

$$|\mathcal{R}_n| = |(F_u \cup F_v \cup F_w \cdots)^c|$$

where the complement is taken in the field $\mathbb{F}_{q^n}$. Using fact (4) we see that $F_u \cap F_v = \mathbb{F}_{q^{n/(uv)}}$ and $F_u \cap F_v \cap F_w = \mathbb{F}_{q^{n/(uvw)}}$, etc.

The cardinality of $\mathcal{R}_n$ can now be computed using the inclusion-exclusion principle as follows.

$$\begin{aligned}|\mathcal{R}_n| = q^n & \\ & - q^{n/u} - q^{n/v} - q^{n/w} - \cdots \\ & + q^{n/uv} + q^{n/uw} + q^{n/vw} + \cdots \\ & \qquad \cdots \\ & + (-1)^r q^{n/uvw\cdots}.\end{aligned}$$

Finally, the formula for $|\mathcal{P}_n|$ takes the desired form when we divide $|\mathcal{R}_n|$ by n and use the Möbius function.

We end by pointing out that if one is interested in counting the cardinality of all irreducible polynomials of degree n (not necessarily monic) over $\mathbb{F}_q$ then we simply multiply $|\mathcal{P}_n|$ by $q-1$. This is because every such polynomial $p(x)$ can be uniquely written as $\alpha r(x)$ where α is a non-zero element of $\mathbb{F}_q$ and $r(x)$ is an irreducible monic polynomial of the same degree.

Acknowledgments Ján Mináč was supported in part by Natural Sciences and Engineering Research Council of Canada grant R3276A01. We would like to thank our students at Illinois State University and University of Western Ontario for their inspiration and insistence on penetrating mysteries of finite fields. We also would like to thank Fusun Akman, Michael Dewar, and Alan Koch for helping us improve the exposition through their interesting and very encouraging comments on this paper.

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Summary C. F. Gauss discovered a beautiful formula for the number of irreducible polynomials of a given degree over a finite field. Using just very basic knowledge of finite fields and the inclusion-exclusion formula, we show how one can see the shape of this formula and its proof almost instantly.

Series Involving Products of Two Harmonic Numbers

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In this paper we prove the identities

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \cdot \frac{H_{n+1}}{n+1} = \frac{\pi^2}{6} + 2\zeta(3) \tag{1}$$

Math. Mag. **84** (2011) 371–377. doi:10.4169/math.mag.84.5.371.

and

$$\sum_{n=1}^{\infty} \frac{H_n}{n} \cdot \frac{H_{n+2}}{n+2} = \frac{5}{4} + \frac{\pi^2}{8} + \zeta(3). \tag{2}$$

These identities link the harmonic numbers, defined for $n \geq 1$ by $H_n = 1 + 1/2 + \cdots + 1/n$, to the Riemann zeta function, defined for integers $k \geq 2$ by $\zeta(k) = 1/1^k + 1/2^k + 1/3^k + \cdots$. Formulas (1) and (2) are the first two cases of our main theorem, below.

Many interesting series involving the harmonic numbers occur in the mathematical literature. Among them is the following classical series formula, due to Euler, which holds for $n \geq 2$:

$$2\sum_{k=1}^{\infty} \frac{H_k}{k^n} = (n+2)\zeta(n+1) - \sum_{k=1}^{n-2} \zeta(n-k)\zeta(k+1). \tag{3}$$

This formula has attracted much interest in recent years and new proofs and other related sums, also evaluated in terms of the Riemann zeta function, have been obtained recently [**1, 3**]. Each term of the series in formula (3) involves a single harmonic number, so it is also known as a *linear Euler sum.*

A *nonlinear* harmonic sum, in contrast, involves products of at least two harmonic numbers. An example of such a series is the following surprising quadratic sum:

$$\sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17\pi^4}{360} = \frac{17}{4}\zeta(4). \tag{4}$$

This identity was discovered numerically by Enrico Au-Yeung, an undergraduate student in the Faculty of Mathematics in Waterloo, and proved rigorously by David Borwein and Jonathan Borwein in [**2**], who proved and used the following integral formula:

$$\int_0^{\pi} \theta^2 \ln^2\left(2\cos\frac{\theta}{2}\right) d\theta = \frac{11\pi^5}{180}.$$

Formula (4) was rediscovered by Freitas as Proposition A.1 in the appendix section of [**5**]. Freitas proved it by calculating a double integral involving a logarithmic function. Motivated by (4), in this paper we investigate the calculation of the following nonlinear harmonic series:

$$S_k = \sum_{n=1}^{\infty} \frac{H_n}{n} \cdot \frac{H_{n+k}}{n+k},$$

where k is a nonnegative integer. Our method, which is elementary, is based on a special integral identity involving the nth harmonic number H_n and some integration techniques involving the Dilogarithm function $\mathrm{Li}_2(x)$.

We prove that, in contrast to the case when $k = 0$, the sum S_k can be written as a rational linear combination of $\zeta(2)$, $\zeta(3)$, and the first k harmonic numbers. The main result of the paper is part (b) of the next theorem. For completeness we include the case when $k = 0$ as part (a).

THEOREM 1.

(a) *The following series identity holds:*

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{17\pi^4}{360}.$$

(b) *Let $k \geq 1$ be a natural number. Then,*

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty}\frac{H_n}{n}\cdot\frac{H_{n+k}}{n+k} \\ &= \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{(j+1)^3}\left(\frac{3}{j+1}-\frac{2}{k}\right) \\ &\quad + \frac{\pi^2}{6}\cdot\frac{H_k}{k} + \frac{2\zeta(3)}{k} - \frac{1}{k}\sum_{i=1}^{k}\frac{H_i}{i^2}. \end{aligned}$$

It is worth mentioning that if k and m are nonnegative integers one can define the following nonlinear harmonic series:

$$T_{k,m} = \sum_{n=1}^{\infty}\frac{H_{n+k}}{n+k}\cdot\frac{H_{n+m}}{n+m}.$$

The sum $T_{k,m}$ can be evaluated using the formula from part (b) of the theorem.

Before we prove Theorem 1 we collect some results that we need in our analysis. Recall that the Dilogarithm function [**7, 8**], denoted by $\text{Li}_2(z)$, is the special function defined for all z satisfying $|z| \leq 1$ by

$$\text{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t}\,dt.$$

A special value of this function is obtained when $z = 1$:

$$\text{Li}_2(1) = \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}.$$

If x is a real number in $(0, 1)$, the following identity holds:

$$\ln x \ln(1-x) - \frac{1}{2}\ln^2(1-x) + \text{Li}_2(x) + \int_1^{1/(1-x)}\frac{\ln(u-1)}{u}\,du = 0. \tag{5}$$

To prove this identity we let $f : (0, 1) \to \mathbb{R}$ be the function defined by the left hand side of (5). Then a straightforward calculation shows that $f'(x) = 0$. Thus, f is a constant function and hence $f(x) = \lim_{x\to 0} f(x) = 0$.

A special integral identity. If m is a nonnegative integer, then

$$\int_0^1 x^m \ln(1-x)\,dx = -\frac{H_{m+1}}{m+1}. \tag{6}$$

This formula, which is quite old, is recorded in various tables of definite integrals. It appears as formula 865.5 in [**4**], where a reference is given to *Nouvelles Tables d'Intégrales Définies*, by B. de Haan: P. Engels, Leyden, 1867, and it also appears, in thin disguise, as entry 4.293(8) of [**6**]. We include below a proof for the sake of completeness. We have,

$$\begin{aligned}\int_0^1 x^m \ln(1-x)\,dx &= \int_0^1 (1-x)^m \ln x\,dx\\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^k\int_0^1 x^k \ln x\,dx\\ &= \sum_{k=0}^{m}\binom{m}{k}(-1)^{k+1}\frac{1}{(k+1)^2}\\ &= \frac{-1}{m+1}\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1}\frac{1}{k+1},\end{aligned}$$

and (6) follows by proving, by induction on m, that

$$\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1}\frac{1}{k+1} = H_{m+1}.$$

Now we are ready to prove the main result of the paper.

Proof of Theorem 1. We have, based on (6), that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n}\cdot\frac{H_{n+k}}{n+k} &= \sum_{n=1}^{\infty}\int_0^1 y^{n-1}\ln(1-y)\,dy\int_0^1 x^{n+k-1}\ln(1-x)\,dx\\ &= \int_0^1\int_0^1 x^k\left(\sum_{n=1}^{\infty}(xy)^{n-1}\right)\ln(1-x)\ln(1-y)\,dx\,dy\\ &= \int_0^1\int_0^1\frac{x^k\ln(1-x)\ln(1-y)}{1-xy}\,dx\,dy\\ &= \int_0^1 x^k\ln(1-x)\left(\int_0^1\frac{\ln(1-y)}{1-xy}\,dy\right)dx.\end{aligned}$$

We calculate the inner integral, by making the substitution $1-xy=t$, and we have

$$\begin{aligned}\int_0^1\frac{\ln(1-y)}{1-xy}\,dy &= \frac{1}{x}\int_{1-x}^1\frac{\ln(1-1/x+t/x)}{t}\,dt\\ &= \frac{1}{x}\int_{1-x}^1\frac{\ln(1/x)}{t}\,dt + \frac{1}{x}\int_{1-x}^1\frac{\ln(x-1+t)}{t}\,dt\\ &= \frac{1}{x}\ln x\ln(1-x) + \frac{1}{x}\int_{1-x}^1\frac{\ln(x-1+t)}{t}\,dt.\end{aligned}$$

The substitution $t = (1-x)u$, in the preceding integral, combined with (5), implies that

$$\begin{aligned}\int_0^1 \frac{\ln(1-y)}{1-xy}\,dy &= \frac{1}{x}\ln x\ln(1-x) + \frac{1}{x}\int_1^{1/(1-x)} \frac{\ln(1-x)(u-1)}{u}\,du \\ &= \frac{1}{x}\left(\ln x\ln(1-x) - \ln^2(1-x) + \int_1^{1/(1-x)} \frac{\ln(u-1)}{u}\,du\right) \\ &= \frac{1}{x}\left(-\frac{1}{2}\ln^2(1-x) - \mathrm{Li}_2(x)\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{H_n}{n}\cdot\frac{H_{n+k}}{n+k} = -\frac{1}{2}\int_0^1 x^{k-1}\ln^3(1-x)\,dx \\ - \int_0^1 x^{k-1}\ln(1-x)\,\mathrm{Li}_2(x)\,dx.\end{aligned} \tag{7}$$

(a) The case $k = 0$. We have, based on (7), that

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = -\frac{1}{2}\int_0^1 \frac{\ln^3(1-x)}{x}\,dx - \int_0^1 \frac{\ln(1-x)\,\mathrm{Li}_2(x)}{x}\,dx.$$

Since $\int_0^1 \left(\ln^3(1-x)/x\right)dx = -\pi^4/15$ (see [**2**, Entry 15, p. 1197]) and $\mathrm{Li}_2(1) = \pi^2/6$, we have

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2 = \frac{\pi^4}{30} + \frac{1}{2}\left(\mathrm{Li}_2(x)\right)^2\Big|_{x=0}^{x=1} = \frac{\pi^4}{30} + \frac{\pi^4}{72} = \frac{17\pi^4}{360}.$$

(b) The case $k \geq 1$. We calculate the first integral of (7) and obtain

$$\begin{aligned}\int_0^1 x^{k-1}\ln^3(1-x)dx &= \int_0^1 (1-x)^{k-1}\ln^3 x\,dx \\ &= \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\int_0^1 x^j\ln^3 x\,dx \\ &= \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{j+1}\frac{6}{(j+1)^4}.\end{aligned} \tag{8}$$

Similarly, one can show that

$$\int_0^1 x^{k-1}\ln^2(1-x)\,dx = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\frac{2}{(j+1)^3}. \tag{9}$$

Now we calculate the second integral of (7) by parts with $f(x) = \mathrm{Li}_2(x)$ and $f'(x) = -\ln(1-x)/x$, $g'(x) = x^{k-1}\ln(1-x)$ and

$$g(x) = \frac{x^k\ln(1-x)}{k} - \frac{1}{k}\left(\ln(1-x) + \frac{x^k}{k} + \frac{x^{k-1}}{k-1} + \cdots + \frac{x^2}{2} + x\right),$$

and we get that

$$\begin{aligned}
&\int_0^1 x^{k-1}\ln(1-x)\,\mathrm{Li}_2(x)\,dx \\
&\quad= \mathrm{Li}_2(x)\left(\frac{(x^k-1)\ln(1-x)}{k} - \frac{1}{k}\sum_{i=1}^{k}\frac{x^i}{i}\right)\Bigg|_{x=0}^{x=1} \\
&\qquad + \frac{1}{k}\int_0^1\left(x^{k-1}\ln^2(1-x) - \frac{\ln^2(1-x)}{x} - \ln(1-x)\sum_{i=1}^{k}\frac{x^{i-1}}{i}\right)dx \\
&\quad= -\mathrm{Li}_2(1)\frac{H_k}{k} + \frac{1}{k}\int_0^1 x^{k-1}\ln^2(1-x)\,dx - \frac{1}{k}\int_0^1\frac{\ln^2(1-x)}{x}dx \\
&\qquad - \frac{1}{k}\sum_{i=1}^{k}\frac{1}{i}\int_0^1 x^{i-1}\ln(1-x)dx \\
&\quad= -\frac{\pi^2}{6}\cdot\frac{H_k}{k} + \frac{1}{k}\int_0^1 x^{k-1}\ln^2(1-x)dx - \frac{2\zeta(3)}{k} + \frac{1}{k}\sum_{i=1}^{k}\frac{H_i}{i^2}, \qquad (10)
\end{aligned}$$

since $\int_0^1\left(\ln^2(1-x)/x\right)dx = 2\zeta(3)$ (see [**6**, Entry 12^7, p. 538]). Combining (7), (8), (9) and (10) we get that

$$S_k = \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{(j+1)^3}\left(\frac{3}{j+1}-\frac{2}{k}\right) + \frac{\pi^2}{6}\cdot\frac{H_k}{k} + \frac{2\zeta(3)}{k} - \frac{1}{k}\sum_{i=1}^{k}\frac{H_i}{i^2},$$

and the theorem is proved. ■

Other interesting series involving the nth harmonic number H_n can be obtained by using formula (6) combined with the calculation of some integrals, simple or multiple, involving logarithmic or Polylogarithm functions. We stop our line of investigation here and we invite the reader to investigate further.

Acknowledgment The author would like to thank Alina Sîntămărian for reading the manuscript and for making valuable suggestions that led to the improvement of the presentation of this paper.

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Summary The paper is about the calculation of the nonlinear harmonic series $S_k = \sum_{n=1}^{\infty} H_n/n \cdot H_{n+k}/(n+k)$, where k is a nonnegative integer and H_n denotes the nth harmonic number. We show that, when $k \geq 1$, the sum S_k can be written as a rational linear combination of $\zeta(2)$, $\zeta(3)$, and the first k harmonic numbers.

"Even with THAT Step Size?"

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The simplest numerical differentiation formula uses the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

to obtain an approximation

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

for small values of $h > 0$. This is known as the forward difference formula. Generally better results can be obtained by the centered difference formula

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}. \tag{1}$$

It is worth noting that the quotient here might have a limit even when the derivative does not exist. For example if $f(x) = |x|$, then for $x_0 = 0$, the right hand side is 0 but there is no derivative at $x_0 = 0$.

Formula (1) can be viewed as a three-point formula of the form

$$f'(x_0) \approx \frac{1}{h}\left[a_{-1} f(x_0 - h) + a_0 f(x_0) + a_1 f(x_0 + h)\right] \tag{2}$$

where $a_{-1} = -1/2$, $a_0 = 0$, and $a_1 = 1/2$ form the weights for the numerical method.

In fact, the centered difference formula (1) is the "best" formula of the form (2) in the sense that the resulting error depends on the highest power of h that can be achieved using the form (2). The corresponding "best" five-point centered difference formula turns out to be

$$f'(x_0) \approx \frac{1}{12h}\left[f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)\right], \tag{3}$$

Math. Mag. **84** (2011) 377–385. doi:10.4169/math.mag.84.5.377. © Mathematical Association of America

where in counting the number of points throughout we realize that the center coefficient of $f(x_0)$ has value 0.

Note that equation (3) can be viewed as a weighted average of centered difference quotients, with weights adding up to one but allowed to be negative:

$$f'(x_0) \approx \frac{4}{3}\left[\frac{f(x_0+h)-f(x_0-h)}{2h}\right] - \frac{1}{3}\left[\frac{f(x_0+2h)-f(x_0-2h)}{4h}\right]$$

Throughout, the formulas we investigate have this kind of weighted average interpretation. A development of these formulae can be found in texts such as [**1**], [**2**], [**10**], and [**12**].

In order to develop a general representation for the optimal formula using $(2N+1)$ points, we consider a numerical differentiation formula of the form,

$$\begin{aligned} f'(x_0) \approx \frac{1}{h}\big[&a_{-N}f(x_0-Nh) + a_{-(N-1)}f(x_0-(N-1)h) + \cdots \\ &+ a_{-2}f(x_0-2h) + a_{-1}f(x_0-h) \\ &+ a_0 f(x_0) \\ &+ a_1 f(x_0+h) + a_2 f(x_0+2h) + \cdots \\ &+ a_{N-1}f(x_0+(N-1)h) + a_N f(x_0+Nh)\big]. \end{aligned} \tag{4}$$

We are assuming that the function $f(x)$ exists and is well defined for every sample point in equation (4).

To obtain the best numerical differentiation formula of this form, we use Taylor expansions:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

We presume that the function has a convergent Taylor series expansion. We replace x with x_0+kh and a with x_0, obtaining (upon changing the order of factors),

$$f(x_0+kh) \approx f(x_0) + (kh)f'(x_0) + \frac{(kh)^2}{2!}f''(x_0) + \frac{(kh)^3}{3!}f'''(x_0) + \cdots$$

where k can be taken positive or negative. To simplify the notation, for fixed h define the right hand side of the equation above by

$$\Delta_k(x_0) \equiv f(x_0) + (kh)f'(x_0) + \frac{(kh)^2}{2!}f''(x_0) + \frac{(kh)^3}{3!}f'''(x_0) + \cdots.$$

We thus expand (4) about $x = x_0$ as follows:

$$\begin{aligned} f'(x_0) \approx \frac{1}{h}[&a_{-N}\Delta_{-N}(x_0) + a_{-(N-1)}\Delta_{-(N-1)}(x_0) + \cdots + a_{-2}\Delta_{-2}(x_0) \\ &+ a_{-1}\Delta_{-1}(x_0) + a_0 f(x_0) + a_1\Delta_1(x_0) \\ &+ a_2\Delta_2(x_0) + \cdots + a_{N-1}\Delta_{(N-1)}(x_0) + a_N\Delta_N(x_0)] \end{aligned} \tag{5}$$

Notice that each term of the form $\Delta_k(x_0)$ contains a coefficient for $f(x_0)$, $f'(x_0)$, etc. To obtain the best possible formula (in the sense described above), we want the right-hand side of formula (5) to approximate $f'(x_0)$ as closely as possible. That is, we want the coefficients of $f(x_0)$ on the right-hand side to add to zero (or drop out), the coefficients of $f'(x_0)$ on the right-hand side to add up to 1 (so that the right-hand side approximates $f'(x_0)$), the coefficients of $f''(x_0)$ on the right-hand side to drop out (to cause the difference between the right-hand side and $f'(x_0)$ to be small), and coefficients of higher order derivatives to continue to drop out to as high order as possible (again, to cause the difference between the right-hand side and $f'(x_0)$ to be small). In other words, we want the error in the approximation to be of highest order in h, thus achieving a formula of optimal formal order of accuracy.

The coefficients

From (5), in order for the coefficients of $f(x_0)$ to drop out we need

$$\begin{aligned} a_{-N} + a_{-(N-1)} + \cdots + a_{-2} + a_{-1} + a_0 + a_1 + a_2 \\ + \cdots + a_{N-1} + a_N = 0. \end{aligned} \tag{6}$$

The coefficients of $f'(x_0)$ on the right need to add up to 1, giving

$$\begin{aligned} -Na_{-N} - (N-1)a_{-(N-1)} - \cdots - 2a_{-2} - a_{-1} + a_1 + 2a_2 \\ + \cdots + (N-1)a_{N-1} + Na_N = 1. \end{aligned} \tag{7}$$

The coefficients of $f''(x_0)$ must drop out giving

$$\begin{aligned} N^2a_{-N} + (N-1)^2a_{-(N-1)} + \cdots + 4a_{-2} + a_{-1} + a_1 + 4a_2 \\ + \cdots + (N-1)^2a_{N-1} + N^2a_N = 0. \end{aligned} \tag{8}$$

Similarly, the coefficients of $f'''(x_0)$ must drop out:

$$\begin{aligned} -N^3a_{-N} - (N-1)^3a_{-(N-1)} - \cdots - 8a_{-2} - a_{-1} + a_1 + 8a_2 \\ + \cdots + (N-1)^3a_{N-1} + N^3a_N = 0. \end{aligned} \tag{9}$$

This pattern continues:

$$\begin{aligned} N^4a_{-N} + (N-1)^4a_{-(N-1)} + \cdots + 16a_{-2} + a_{-1} + a_1 + 16a_2 \\ + \cdots + (N-1)^4a_{N-1} + N^4a_N = 0, \\ -N^5a_{-N} - (N-1)^5a_{-(N-1)} - \cdots - 32a_{-2} - a_{-1} + a_1 + 32a_2 \\ + \cdots + (N-1)^5a_{N-1} + N^5a_N = 0, \\ \vdots \\ N^{2N}a_{-N} + (N-1)^{2N}a_{-(N-1)} + \cdots - 2^{2N}a_{-2} + a_{-1} + a_1 + 2^{2N}a_2 \\ + \cdots + (N-1)^{2N}a_{N-1} + N^{2N}a_N = 0. \end{aligned} \tag{10}$$

Consider more closely the seven equations from (6) through (10). They form a linear system in the variables $a_{-N}, a_{-(N-1)}, \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots, a_{N-1}, a_N$. The coefficient matrix for this $(2N+1)$-by-$(2N+1)$ linear system is a Vandermonde matrix, which is known to be invertible. Thus, the system has a unique solution. In fact, the solution is known to be given by coefficients of Lagrange polynomials. Here, though, we take a different approach.

Looking back to formula (4), since the best $(2n+1)$-point formula is unique, we must obtain the same formula whether we work forwards with $h > 0$ or work backwards by replacing h by $-h$. For example, replacing h by $-h$ in our simplest centered numerical differentiation formula (1), we obtain

$$f'(x_0) \approx \frac{f(x_0-h)-f(x_0+h)}{-2h} = \frac{f(x_0+h)-f(x_0-h)}{2h}$$

Similarly, replacing h by $-h$ in (5),

$$\begin{aligned} f'(x_0) \approx \frac{1}{-h}[&a_{-N}\Delta_N(x_0) + a_{-(N-1)}\Delta_{(N-1)}(x_0) + \cdots + a_{-2}\Delta_2(x_0) \\ &+ a_{-1}\Delta_1(x_0) + a_0 f(x_0) + a_1\Delta_{-1}(x_0) \\ &+ a_2\Delta_{-2}(x_0) + \cdots + a_{N-1}\Delta_{-(N-1)}(x_0) + a_N\Delta_{-N}(x_0)] \\ \approx \frac{1}{h}[&-a_N\Delta_{-N}(x_0) - a_{(N-1)}\Delta_{-(N-1)}(x_0) - \cdots - a_2\Delta_{-2}(x_0) \\ &- a_1\Delta_{-1}(x_0) - a_0 f(x_0) - a_{-1}\Delta_1(x_0) \\ &- a_{-2}\Delta_2(x_0) - \cdots - a_{-(N-1)}\Delta_{(N-1)}(x_0) - a_N\Delta_N(x_0)] \end{aligned} \tag{11}$$

Comparing (11) with (5), we find that $a_{-k} = -a_k$ for all k, and so also $a_0 = 0$. Speaking very informally, if we had a "lopsided" solution in which $|a_{-k}| > |a_k|$ (or $|a_{-k}| < |a_k|$) for some k, then we would have a second "lopsided" solution in which $|a_{-k}| < |a_k|$ (or $|a_{-k}| > |a_k|$) for that k.

We now further examine the equations in (6)–(10), replacing a_{-k} by $-a_k$ and setting $a_0 = 0$. Equation (6) is now satisfied automatically. Equation (7) becomes

$$\sum_{k=1}^{N} k a_k = \frac{1}{2}.$$

Equation (8) is now satisfied automatically. Equation (9) becomes

$$\sum_{k=1}^{N} k^3 a_k = 0.$$

Generalizing, every equation involving even powers of N is now satisfied automatically. We thus obtain the following linear system in $a_1, a_2, \ldots, a_N$:

$$\sum_{k=1}^{N} k a_k = \frac{1}{2}, \quad \sum_{k=1}^{N} k^3 a_k = 0, \quad \ldots, \quad \sum_{k=1}^{N} k^{2N-1} a_k = 0, \tag{12}$$

where we have N equations in the variables $a_1, \ldots, a_N$. In matrix form, this system reads

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & N \\ 1^3 & 2^3 & 3^3 & \cdots & N^3 \\ & & \vdots & & \\ 1^{2N-1} & 2^{2N-1} & 3^{2N-1} & \cdots & N^{2N-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{13}$$

Note that we have reduced what was a $(2N+1)$-by-$(2N+1)$ system down to an N-by-N system. This reduction is achieved through using the symmetrical properties of the centered difference formula and the uniqueness of the "best" formula.

Solving system (13) with $N = 1$ and with $N = 2$, we retrieve the 3-point and 5-point formulas given earlier in equations (1) and (3), where as before in counting the number of points throughout we realize that the coefficient of $f(x_0)$ has value 0.

With $N = 3$, the solution to the system is $a_1 = 3/4$, $a_2 = -3/20$, $a_3 = 1/60$, giving the 7-point formula,

$$f'(x_0) \approx \frac{1}{60h}[-f(x_0 - 3h) + 9f(x_0 - 2h) - 45f(x_0 - h) + 45f(x_0 + h) - 9f(x_0 + 2h) + f(x_0 + 3h)]. \tag{14}$$

With $N = 4$, the solution to the system is $a_1 = 4/5$, $a_2 = -1/5$, $a_3 = 4/105$, $a_4 = -1/280$, giving the 9-point formula,

$$f'(x_0) \approx \frac{1}{840h}[3f(x_0 - 4h) - 32f(x_0 - 3h) + 168f(x_0 - 2h) - 672f(x_0 - h) + 672f(x_0 + h) - 168f(x_0 + 2h) + 32f(x_0 + 3h) - 3f(x_0 + 4h)]. \tag{15}$$

The number pattern

When we looked at solutions to system (13) for larger values of N, we obtained by inspection the following solution for general N:

$$a_1 = \frac{N}{N+1}, \quad a_2 = -\frac{N(N-1)}{2(N+1)(N+2)}, \tag{16}$$

and in general for $k = 1, \ldots, N$,

$$a_k = \frac{(-1)^{k+1}}{k} \cdot \frac{\frac{N!}{(N-k)!}}{\frac{(N+k)!}{N!}} = \frac{(-1)^{k+1}}{k} \cdot \frac{\binom{N}{k}}{\binom{N+k}{k}}. \tag{17}$$

The formula for a_k may be rewritten as

$$a_k = \frac{(-1)^{k+1}}{k} \cdot \frac{(N!)^2}{(N+k)!\,(N-k)!}.$$

In this form, we can see that $a_{-k} = -a_k$ by replacing k with $-k$.

Values for these coefficients have been established since the middle of the twentieth century when much interest was given to formulating finite difference schemes to approximate derivatives. These coefficients, or weights, are typically developed using an interpolating polynomial having the appropriate values at distinct nodal (or mesh) points. Determining appropriate weights for these approximations usually involves solving a Vandermonde linear system (see [**6**], [**8**], [**9**] and [**14**]). The system equation in (13) is almost a Vandermonde system since it has a similar structure but is simpler and of lower order. The difference in the system structures reflects the approach of using symmetry instead of an interpolating polynomial.

As one might suspect, information gathered from far away nodes will be weighted less than those nearby. Tables illustrating the values of these weights can be found in many sources such as [**4**], [**7**], and [**10**]. Fast algorithms can generate the coefficient weights for all order of derivatives (for example, [**3**], [**13**], and [**15**]).

We mentioned above that we found the coefficients in (16) and (17) by inspection. Informally speaking, we "discovered" the combinatorial expressions in (16) and (17) partly through trial and error and by looking at the number patterns for small values N. This approach may not seem sensible, but can often provide valuable information such as was discovered here.

The plots in FIGURE 1 show the values of a_k and a_{-k} (with $a_{-k} = -a_k$) for $N = 10$ and for $N = 50$. As $N \to \infty$, we have that $a_1 \to 1$, $a_2 \to -1/2$, and in fact $a_k \to (-1)^{k+1}/k$.

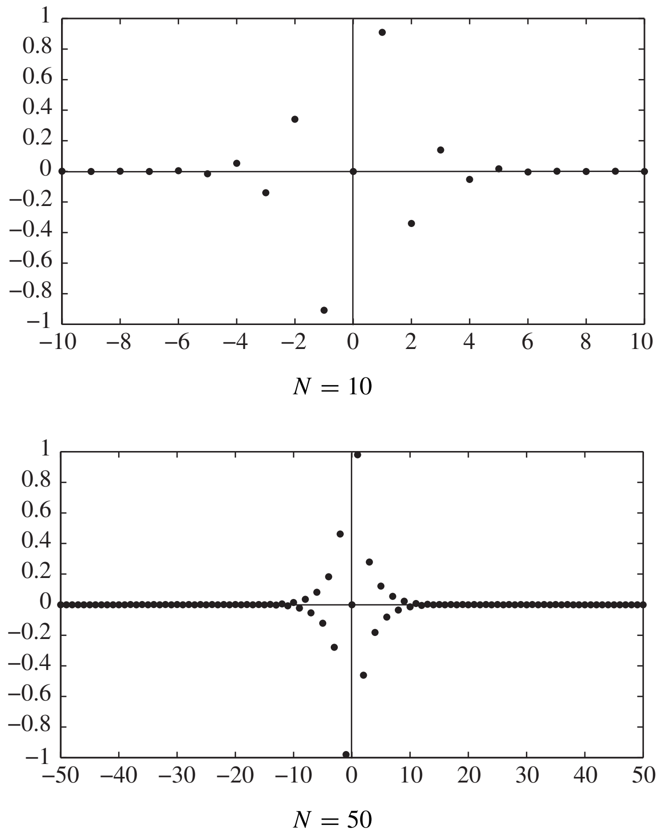

Figure 1 Values of a_k and a_{-k} for $N = 10$ and for $N = 50$.

We have thus obtained the following form for our $(2N+1)$-point numerical differentiation formula:

$$
\begin{aligned}
f'(x_0) \approx \frac{1}{h}\Bigg[&-\frac{(-1)^{N+1}}{N} \cdot \frac{1}{\binom{2N}{N}} f(x_0 - Nh) - \frac{(-1)^N}{N-1} \cdot \frac{N}{\binom{2N-1}{N-1}} f(x_0 - (N-1)h) \\
&+ \cdots + \frac{N(N-1)}{2(N+1)(N+2)} f(x_0 - 2h) - \frac{N}{N+1} f(x_0 - h) \\
&+ \frac{N}{N+1} f(x_0 + h) - \frac{N(N-1)}{2(N+1)(N+2)} f(x_0 + 2h) + \cdots \qquad (18) \\
&+ \frac{(-1)^N}{N-1} \cdot \frac{N}{\binom{2N-1}{N-1}} f(x_0 + (N-1)h) + \frac{(-1)^{N+1}}{N} \cdot \frac{1}{\binom{2N}{N}} f(x_0 + Nh) \Bigg]
\end{aligned}
$$

where we have used the fact that $\binom{N}{N} = 1$ and $\binom{N}{N-1} = N$.

Examples

To illustrate the precision of the numerical method we have used five sample functions for comparison purposes. Although numerical methods are not needed for these functions, they are well suited for comparing our methods with analytical results. Given in TABLE 1 is the computed derivative for the five functions (with a corresponding x_0-value) for various step sizes and number of nodes. The computed approximation of $f'(x_0)$ is given in double precision 14 or 15 digit decimal expansion. Also illustrated in TABLE 1 is the corresponding absolute error for the numerical method.

TABLE 1: This table shows the numerical method applied for different functions using different step sizes and different number of nodes. Also included is the absolute error between the "true value" and the numerical approximation of the first derivative.

$f(x), x_0$	Step Size	N	Approximation $f'(x_0)$	Abs. Error
$\sin x$,	$h = 0.1$	$N = 5$	0.070737201667701	2.4×10^{-15}
$x_0 = 1.5$ (radians)		$N = 10$	0.070737201667703	5.55×10^{-17}
	$h = 0.5$	$N = 5$	0.070737178851280	2.28×10^{-8}
True value ≈		$N = 10$	0.070737201667689	1.43×10^{-14}
0.070737201667703	$h = 1$	$N = 5$	0.070719283094854	1.79×10^{-5}
		$N = 10$	0.070737193179064	8.49×10^{-9}
e^x,	$h = 0.1$	$N = 5$	7.389056098930924	2.74×10^{-13}
$x_0 = 2$		$N = 10$	7.389056098930656	5.33×10^{-15}
	$h = 0.5$	$N = 5$	7.389058941731766	2.84×10^{-6}
True value ≈		$N = 10$	7.389056098928454	2.20×10^{-12}
7.389056098930650	$h = 1$	$N = 5$	7.392844411006770	3.79×10^{-3}
		$N = 10$	7.389052038082562	4.06×10^{-6}
$\ln x$,	$h = 0.01$	$N = 5$	0.399999999999990	1.03×10^{-14}
$x_0 = 2.5$		$N = 10$	0.399999999999977	2.26×10^{-14}
	$h = 0.05$	$N = 5$	0.400000000000000	1.11×10^{-16}
True value:		$N = 10$	0.399999999999995	5.22×10^{-15}
0.4	$h = 0.1$	$N = 5$	0.400000000005922	5.92×10^{-12}
		$N = 10$	0.400000000000000	1.11×10^{-16}
$4x^3 - 2x^2 + x - 7$,	$h = 0.1$	$N = 5$	97	0
$x_0 = 3$		$N = 10$	97.000000000000043	4.26×10^{-14}
	$h = 0.5$	$N = 5$	97.000000000000014	1.42×10^{-14}
True value:		$N = 10$	97	0
97.0	$h = 1$	$N = 5$	97.000000000000014	1.42×10^{-14}
		$N = 10$	97	0
$\arctan x$,	$h = 0.1$	$N = 5$	0.075471698113213	5.09×10^{-15}
$x_0 = 3.5$		$N = 10$	0.075471698113206	1.85×10^{-15}
	$h = 0.5$	$N = 5$	0.075470707677942	9.9×10^{-7}
True value ≈		$N = 10$	0.075461995697012	9.7×10^{-6}
0.075471698113208	$h = 1$	$N = 5$	0.077567342968624	2.1×10^{-3}
		$N = 10$	0.073121415093676	2.35×10^{-3}

Notice that for these functions, even a relatively large step size can give a very good approximation to the actual value of the derivative at x_0. We find it interesting that these formulas work at all for large N-values with fairly large h-values. It is remarkable, for instance, to look at the results that are shown with $N = 10$ and step size $h = 1$. In these cases, we are doing function evaluations up to a full unit to the left and right of x_0 and still getting very accurate results. It is important to note that issues pertaining to domains, discontinuities, undefined derivatives, etc., have not been addressed here.

The approximations given in TABLE 1 appear to get better as h gets smaller leading the reader to believe that the error becomes smaller as h goes to zero. Unfortunately, this is not always the case and the error between the true value and the approximation can be very sensitive to the limitations of computer arithmetic in the calculation of the function values, especially for very small h. Alternatively, choosing large values of h can lead to large errors in the Taylor approximations. Thus, one has to be aware of the potential hazards when using numerical differentiation formulas. You may have to choose between potentially large theoretical errors if h is too big or potentially large computational errors if h is too small.

You may note that in several cases, the $N = 10$ results are not as good as the $N = 5$ results. We believe that there are a number of computational issues coming into play here, including rounding error, the interval length over which we are doing function evaluations, the number of function evaluations, the behavior of the specific functions being evaluated, etc. Whenever one moves from theory to computational work, these kinds of issues come into play.

Many other ideas have been left for the interested reader to explore. For example, what happens to the approximation as $N \to \infty$? To answer this question requires looking at the idea of an "infinite point" numerical differentiation formula. Such a formula would have the form given by

$$f'(x_0) \approx \frac{1}{h}\left[\cdots - \frac{1}{3}f(x_0 - 3h) + \frac{1}{2}f(x_0 - 2h) - f(x_0 - h) + f(x_0 + h) - \frac{1}{2}f(x_0 + 2h) + \frac{1}{3}f(x_0 + 3h) - \cdots\right]$$

When investigating such an approximation, issues concerning domains, convergence, and validity quickly come to mind. It would be interesting to study the properties of such a numerical differentiation formula and to determine a class of functions to which they apply.

Finally, we highlight a few mathematical formulas that emerged while developing our approximation. Note that the first equation in (12) gives the identity

$$\sum_{k=1}^{N} 2(-1)^{k+1} \cdot \frac{\binom{N}{k}}{\binom{N+k}{k}} = 1.$$

Although a combinatorial proof for this identity exists, we simply point out that we can achieve this result from a purely numerical construction. Similarly, again from (12), for $0 < r \leq 2N - 2$, with r even-valued, we have

$$\sum_{k=1}^{N} (-1)^{k+1} \cdot \frac{\binom{N}{k}}{\binom{N+k}{k}} \cdot k^r = 0.$$

These formulas (or general versions of them) can be found in [**5**], where a list of frequently occurring and interesting combinatorial identities has been given.

Not only do numerical differentiation schemes have value in approximating derivatives, they can be used to develop and formulate many interesting number patterns!

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Summary The centered difference quotient is the simplest centered approximation for the derivative of a function at a point. In this paper, we explore and apply general centered approximation formulae for the derivative of a function at a point using any number of weighted evenly-spaced nodes. The resulting formulae have connections to both Vandermonde matrices and Lagrange polynomials and lead to interesting combinatorial identities.

PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by May 1, 2012.

1881. *Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia.*

Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ a convex function (equivalently a concave upward function). Define

$$F(s, t) = f(s) + f(t) - 2f\left(\frac{s+t}{2}\right).$$

Prove that $F(s, t) \leq F(a, b)$ for every $s, t \in [a, b]$.

1882. *Proposed by Timothy Hall, PQI Consulting, Cambridge, MA.*

Prove that

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \frac{3}{4}\pi$$

and find an error estimate for the partial sums.

1883. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

The lengths of the sides of a plane hexagon *ABCDEF* (not necessarily convex) satisfy that $2AB = BC$, $2CD = DE$, and $2EF = FA$. Prove that

$$\frac{AF}{CF} + \frac{CB}{EB} + \frac{ED}{AD} \geq 2.$$

Math. Mag. **84** (2011) 386–393. doi:10.4169/math.mag.84.5.386.

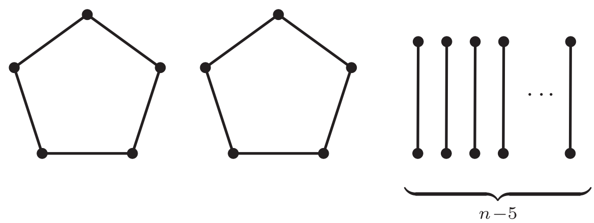
We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1884. *Proposed by Khodakhast Bibak, Department of Combinatorics & Optimization, University of Waterloo, Waterloo, Canada.*

Let $n \geq 5$ be a positive integer. Consider the graphs G on $2n$ vertices that are triangle-free and have independence number $\alpha(G) < n$. Prove that if G has the minimum possible number of edges, then G is isomorphic to the disjoint union of two cycles of length 5 and $n - 5$ edges.



1885. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let $n \geq 1$ be an integer and denote by $\lfloor x \rfloor$ the integer part of x. Let $x_1, \ldots, x_n \geq 1$ be real numbers such that

$$\sum_{j=1}^{n} \lfloor x_j^k \rfloor = \sum_{j=1}^{n} \lfloor x_j \rfloor^k$$

is true for infinitely many positive integers k. Prove that all the numbers $x_1, \ldots, x_n$ are integers.

Quickies

Answers to the Quickies are on page 393.

Q1015. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let f be a strictly increasing, but not necessarily continuous, real valued function defined on the closed interval $[a, b]$. Let the function $g : [f(a), f(b)] \to [a, b]$ be defined as $g(y) = \sup\{t \in [a, b] : f(t) \leq y\}$ for each $y \in [f(a), f(b)]$.

1. Is g onto?
2. Is g continuous on $[f(a), f(b)]$?

Q1016. *Proposed by Herman Roelants, Center for Logic, Institute of Philosophy, University of Louvain, Leuven, Belgium.*

Which integer powers of 2 cannot be written in the form $ab + (a + 1)(b + 1)$ for a and b positive integers?

Solutions

Different row sums and equal column sums **December 2010**

1856. *Proposed by H. A. ShahAli, Tehran, Iran.*

(a) Determine all the positive integers n for which there exists an $n \times n$ array of entries in $\{0, 1\}$ such that the row sums are pairwise distinct and the column sums are all equal.

(b) Determine all such positive integers n under the additional restriction that every row has at least one entry equal to 1.

Solution by Eddie Cheng and Jerrold W. Grossman, Oakland University, Rochester, MI.

For part (a) it is always possible. If n is odd, the nth row has all ones, and for $k = 1, 2, \ldots, (n-1)/2$ the $(2k-1)$th row has ones in the first k positions and zeros in the last $n-k$ positions, whereas the $(2k)$th row has zeros in the first k positions and ones in the last $n-k$ positions. For example, for $n = 5$ we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The row sums are $1, 2, \ldots, n$, and the column sums are all $(n+1)/2$.

The solution for even n is similar. For $k = 0, 1, 2, \ldots, (n-2)/2$ the $(2k+1)$th row has ones in the first k positions and zeros in the last $n-k$ positions, whereas the $(2k+2)$th row has zeros in the first k positions and ones in the last $n-k$ positions. For example, for $n = 6$ we have

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The row sums are all the integers from 0 to n except $n/2$, and the column sums are all $n/2$.

For part (b) note that the solution given above for odd n satisfies the extra condition. Furthermore, no such array exists for even n. Indeed, in that case, the row sums would have to be $1, 2, \ldots, n$, so there would be $n(n+1)/2$ ones altogether in the array. This would imply that each column sum would have to be $(n+1)/2$, which is not an integer.

Also solved by Armstrong Problem Solvers, Michael R. Bacon and Charles K. Cook, Mark Bowron, Cal Poly Pomona Problem Solving Group, Douglas Cashing, Con Amore Problem Group, Calvin A. Curtindolph, Chip Curtis, Daniele Degiorgi (Switzerland), Patrick Devlin, Dave Feil, Natacha Fontes-Merz, Fullerton College Mathematics Association, David Getling (Germany), Michael Goldenberg and Mark Kaplan, Laura Iosip (United Kingdom), Iowa State Problem Solving Group, Omran Kouba (Syria), Victor Y. Kutsenok, Elias Lampakis (Greece), Kathleen E. Lewis (Republic of the Gambia), Reiner Martin (Germany), Peter McPolin (Northern Ireland), Shoeleh Mutameni, David Nacin, José Heber Nieto (Venezuela), Northwestern University Math Problem Solving Group, Rob Pratt, David Roberts, Joel Schlosberg, Randy K. Schwartz, Skidmore College Problem Group, Philip Straffin, Jeff Stuart, Marian Tetiva (Romania), Texas State University Problem Solving Group, and the proposer. There was one incomplete solution and there was one incorrect solution.

Bounding the area of an inscribed triangle December 2010

1857. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania; and Tudorel Lupu, Decebal High School, Constanta, Romania*

Let ABC be an arbitrary triangle with $a = BC$, $b = AC$, and $c = AB$. The points A_1, B_1, and C_1, on the segments BC, AC, and AB, respectively, satisfy that $AB + BA_1 = AC + CA_1$, $BC + CB_1 = BA + AB_1$, and $CA + AC_1 = CB + BC_1$. Prove that

$$\frac{\text{Area}(A_1B_1C_1)}{\text{Area}(ABC)} \le \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)}.$$

Solution by Michel Bataille, Rouen, France.

Let $F = \text{Area}(ABC)$ and $F_1 = \text{Area}(A_1B_1C_1)$ and let $s = (a+b+c)/2$ be the semi-perimeter of the triangle ABC. Let $x = BA_1$, by the condition $AB + BA_1 = AC + CA_1$, we get that $c + x = b + a - x$, that is, $BA_1 = x = (b+a-c)/2 = s - c$; now since $BA_1 + CA_1 = a$, we obtain also that $CA_1 = s - b$. Similarly, we have $CB_1 = s - a$, $AB_1 = s - c$, $AC_1 = s - b$, and $BC_1 = s - a$. Using these identities, we have that

$$\text{Area}(AB_1C_1) = \frac{(s-b)(s-c)\sin\angle BAC}{2} = (s-b)(s-c)\cdot\frac{F}{bc}.$$

Similarly,

$$\text{Area}(A_1BC_1) = (s-a)(s-c)\frac{F}{ac}$$

and

$$\text{Area}(A_1B_1C) = (s-a)(s-b)\frac{F}{ab}.$$

Since $F_1 = F - \text{Area}(AB_1C_1) - \text{Area}(A_1BC_1) - \text{Area}(A_1B_1C)$, we obtain that

$$F_1 = F - F\left(\frac{(s-b)(s-c)}{bc} + \frac{(s-a)(s-c)}{ac} + \frac{(s-a)(s-b)}{ab}\right),$$

and then

$$\begin{aligned}\frac{abcF_1}{F} &= abc - a(s-b)(s-c) - b(s-a)(s-c) - c(s-a)(s-b)\\ &= 2(s-a)(s-b)(s-c).\end{aligned}$$

Therefore the required inequality is equivalent to

$$8(s-a)(s-b)(s-c)\cdot 2s\cdot(a^2+b^2+c^2) \le 9(abc)^2.$$

Because $s(s-a)(s-b)(s-c) = F^2$ and $4RF = abc$, where R denotes the circumradius of the triangle ABC, it follows that the last inequality is equivalent to $a^2 + b^2 + c^2 \le 9R^2$. This well-known inequality follows from Leibniz's Identity $OG^2 = R^2 - (a^2+b^2+c^2)/9$, where O and G are the circumcenter and the barycenter of ABC, respectively.

Editor's Note. Minh Can, Omran Kouba, and Peter Nüesch (independently) note that the left side of the inequality is equal to $r/2R$ where r and R are the inradius and the circumradius, respectively, of the triangle ABC.

Also solved by Herb Bailey; Robert Calcaterra; Minh Can; Chip Curtis; Daniele Degiorgi (Switzerland); John G. Heuver (Canada); Enkel Hysnelaj (Australia) and Elton Bojaxhiu (Germany); Omran Kouba (Syria); Richard Lopez; José H. Nieto (Venezuela); Peter Nüesch (Switzerland); Scott Pauley, Natalya Weir, and Andrew Welter; Joel Schlosberg; Michael Vowe (Switzerland); Haohao Wang and Jerzy Woydylo, and the proposers. There were two incorrect submissions.

Fermat's Theorem resurrected! **December 2010**

1858. *Proposed by Herman Roelants, Center for Logic, Institute of Philosophy, University of Louvain, Leuven, Belgium.*

Let $p \geq 3$ be an odd integer. Prove that the equation $u^p + 4^{p-1} = v^2$ has rational solutions (u, v) with $uv \neq 0$, if and only if, the equation $x^p + y^p = z^p$ has integer solutions (x, y, z) with $xyz \neq 0$.

Solution by Stefan Chatadus, Poland.
We modified the statement of the problem to avoid the confusion related to the term "nonzero solution". With the usual understanding of this term, the problem is trivial since $(0, \pm 2^{p-1})$ are nonzero solutions of the first Diophantine equation and $(0, 1, 1)$ and $(0, -1, -1)$ are nonzero solutions of the second equation. Also, let us observe that it makes no difference if we require the existence of nontrivial solutions or just one solution.

For sufficiency, if (x, y, z) is a nontrivial solution of $x^p + y^p = z^p$, without loss of generality we may assume that y is a positive even number, i.e., $y = 2t$, $t \in \mathbb{N}$. We observe that $4x^pz^p + (z^p - x^p)^2 = (x^p + z^p)^2$ and so $4x^pz^p + 4^pt^{2p} = (x^p + z^p)^2$. Dividing by $4t^{2p}$ the last equality, we obtain

$$\left(\frac{xz}{t^2}\right)^p + 4^{p-1} = \left(\frac{x^p + z^p}{2t^p}\right)^2$$

with $xz \neq 0$ and $x^p + z^p = 2x^p + y^2 \neq 0$.

In order to show the necessity, let u and v be two nonzero rational numbers satisfying $u^p + 4^{p-1} = v^2$. Without loss of generality we may write $u = a/b$, $v = c/d$, with nonzero integers a, b, c, and d such that $b > 0$, $d > 0$ and $\gcd(a, b) = \gcd(c, d) = 1$.

We have the two equivalent forms of the equation $(a/b)^p + 4^{p-1} = (c/d)^p$:

$$d^2(a^p + 4^{p-1}b^p) = b^pc^2 \text{ and } d^2a^p = b^p(c^2 - 4^{p-1}d^2).$$

Since $\gcd(d^2, c^2) = 1$ we see that d^2 must divide b^p. Also, because $\gcd(b^p, a^p) = 1$, from the last equality, we see that b^p must divide d^2. Thus

$$d^2 = b^p \text{ and } a^p + 4^{p-1}b^p = c^2.$$

Given that p is odd, examining the prime factorizations of d and b one sees that there exists a positive integer e such that and $b = e^2$ and $d = e^p$. Clearly, e and c are relatively prime. Thus

$$a^p + 4^{p-1}e^{2p} = c^2 \Leftrightarrow a^p = (c - 2^{p-1}e^p)(c + 2^{p-1}e^p).$$

Let $q = \gcd(c - 2^{p-1}e^p, c + 2^{p-1}e^p)$. Consider two cases.

CASE I: $q = 1$. The last equality implies there are integers f and g such that

$$c - 2^{p-1}e^p = f^p \text{ and } c + 2^{p-1}e^p = g^p.$$

This gives

$$(2e)^p + f^p = g^p.$$

Since $a \neq 0$, f and g are also nonzero, and because $b \neq 0$ then $e \neq 0$. Hence, the obtained solution has no zero components.

CASE II: $q > 1$. Suppose r is a prime divisor of q. We must have r a divisor of both 2^pe^p and $2c$. We observe that r cannot divide e because that leads to the contradiction that r is a divisor of $\gcd(a, e^2) = \gcd(a, b) = 1$. Hence, we conclude that $\gcd(q, e) = 1$ and $q = 2^m$ for some integer m, $1 \le m \le p$, with e an odd integer. Let us observe that $a^p = (c - 2^{p-1}e^p)(c + 2^{p-1}e^p)$ implies that 2^{2m} divides a^p, and $2m$ is the highest exponent with this property. Our assumption about p as being odd implies that $a = 4a'$ for some integer a'. This shows that $m = p$ and $c = 2^{p-1}c'$ for some odd integer c'. The equation we get now is $4a'^p = (c' - e^p)(c' + e^p)$ with $\gcd(c' + e^p, c' - e^p) = 2$. As we have seen before, in the case $q = 1$, this leads to the existence of integers f and g such that

$$\frac{c' - e^p}{2} = f^p \text{ and } \frac{c' + e^p}{2} = g^p.$$

Therefore $g^p - f^p = e^p$ and the proof is complete.

Editor's Note. Also mentioned by S. Chatadus, the problem follows from Theorem 1 M (Ch. VIII) in the book "Fermat's Last Theorem for Amateurs" written by P. Ribemboim and published in 1999.

Also solved by George Apostolopulous (Greece) and the proposer. There were five incorrect submissions.

Integral manipulation. December 2010

1859. *Proposed by Valmir Krasniqi, Department of Mathematics, University of Prishtina, Prishtinë, Republic of Kosova.*

Let α be a positive real number and f be a nonnegative function on [0, 1] such that

$$\int_x^1 (f(t))^\alpha \, dt \ge \int_x^1 t^\alpha \, dt \text{ for all } 0 \le x \le 1.$$

Prove that

$$\int_0^1 (f(t))^{\alpha+\beta} \, dt \ge \int_0^1 (f(t))^\alpha t^\beta \, dt \ge \int_0^1 t^{\alpha+\beta} \, dt$$

for every positive real β.

Solution by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.

First we prove the second inequality. Reversing the order of integration and using the given condition gives

$$\begin{aligned}
\int_0^1 (f(t))^\alpha \, t^\beta \, dt &= \frac{1}{\beta} \int_0^1 (f(t))^\alpha \left(\int_0^t u^{\beta-1} \, du \right) dt \\
&= \frac{1}{\beta} \int_0^1 u^{\beta-1} \left(\int_u^1 (f(t))^\alpha \, dt \right) du \\
&\ge \frac{1}{\beta} \int_0^1 u^{\beta-1} \left(\int_u^1 t^\alpha \, dt \right) du \\
&= \frac{1}{\beta} \int_0^1 t^\alpha \left(\int_0^t u^{\beta-1} \, du \right) dt \\
&= \int_0^1 t^{\alpha+\beta} \, dt,
\end{aligned}$$

and this proves the second inequality.

The weighted version of the Arithmetic Mean–Geometric Mean Inequality implies that

$$\frac{\alpha}{\alpha+\beta}(f(t))^{\alpha+\beta}+\frac{\beta}{\alpha+\beta}t^{\alpha+\beta}\geq (f(t))^{\alpha}\, t^{\beta}.$$

Integrating both sides, since the function is nonnegative and using the inequality proved before gives

$$\begin{aligned}\frac{\alpha}{\alpha+\beta}\int_0^1 (f(t))^{\alpha+\beta}\, dt &\geq \int_0^1 (f(t))^{\alpha}\, t^{\beta}\, dt-\frac{\beta}{\alpha+\beta}\int_0^1 t^{\alpha+\beta}\, dt\\ &\geq \int_0^1 (f(t))^{\alpha}\, t^{\beta}\, dt-\frac{\beta}{\alpha+\beta}\int_0^1 (f(t))^{\alpha}\, t^{\beta}\, dt\\ &=\frac{\alpha}{\alpha+\beta}\int_0^1 (f(t))^{\alpha}\, t^{\beta}\, dt,\end{aligned}$$

and the first inequality follows after dividing by $\alpha/(\alpha+\beta)$.

Also solved by Michael W. Botsko, Omran Kouba (Syria), Peter W. Lindstrom, Duong Viet Thong (Vietnam), Haohao Wang and Yanping Xia, and the proposer.

Finding roots on the unit circle **December 2010**

1860. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu", Bârlad, Romania.*

Let α be a complex number such that $|\alpha|>1$ and let n be an integer such that $n>2$. Prove that at least $n-2$ roots of the equation $z^n+\alpha z^{n-1}+\overline{\alpha}z+1=0$ have norm equal to 1.

Solution by George Apostolopoulos, Messolonghi, Greece.

Because $|\alpha|\neq 1$, it follows that $z=-1/\overline{\alpha}$ is not a root of the given equation. Clearly $z=0$ is not a solution either, thus the given equation is equivalent to

$$\frac{z+\alpha}{\overline{\alpha}z+1}=-\frac{1}{z^{n-1}}.$$

If $|z|=1$, then $|z+\alpha|=|\overline{z}||z+\alpha|=|\alpha\overline{z}+1|=|\overline{\alpha}z+1|$ and $|z^{n-1}|=1$. It follows that the functions $(z+\alpha)/(\overline{\alpha}z+1)$ and $-1/z^{n-1}$ map the unit circle $|z|=1$ to itself. Let $z=\cos\theta+i\sin\theta$. As θ increases from 0 to 2π, the point $(z+\alpha)/(\overline{\alpha}z+1)$ goes around the circle $|z|=1$ clockwise for one lap. While the point $-1/z^{n-1}$ goes around the same circle clockwise for $n-1$ laps. One can imagine these two functions as runners running on a circular track. They run in the same direction for one lap and $n-1$ laps respectively. Thus, regardless of their initial position, they will meet at least $n-2$ times. Therefore, there are at least $n-2$ angles θ in the interval $[0,2\pi)$ such that $z=\cos\theta+i\sin\theta$ is a solution to the given equation.

Editor's Note. If $|\alpha|=1$, then the given equation is equivalent to $(z+\alpha)(z^{n-1}+1/\alpha)=0$ which has n roots of norm equal to 1. Some readers pointed out that all the roots have norm equal to 1 whenever $|\alpha|>1$.

Also solved by Armstrong Problem Solvers, G.R.A.20 Problem Solving Group (Italy), Omran Kouba (Syria), Kee-Wai Lau (China), Kim McInturff, Peter McPolin (Northern Ireland), Raymond Mortini (France), José Heber Nieto (Venezuela), Nothwestern University Math Problem Solving Group, and the proposer. There was one incomplete solution.

Answers

Solutions to the Quickies from page 387.

A1015.

1. The function g is onto. To see this we show that $g(f(x)) = x$ for all x in $[a, b]$. Take $x \in [a, b]$ and consider $g(f(x)) = \sup\{t \in [a, b] : f(t) \leq f(x)\}$. Since $x \in \{t \in [a, b] : f(t) \leq f(x)\}$, we have that $g(f(x)) \geq x$. Now for any $t_0 \in \{t \in [a, b] : f(t) \leq f(x)\}$, we have that $t_0 \leq x$. This is true because if $t_0 > x$, then $f(t_0) > f(x)$ which is a contradiction. Therefore $g(f(x)) \leq x$ so that $g(f(x)) = x$ for all x in $[a, b]$. Thus $g : [f(a), g(b)] \to [a, b]$ is onto.
2. The function g is also continuous on $[f(a), f(b)]$. To see this first note that g is clearly non-decreasing on $[f(a), f(b)]$. Let us now take $y_0 \in (f(a), f(b))$. Since g is non-decreasing,

$$\lim_{y \to y_0^-} g(y) \leq g(y_0) \leq \lim_{y \to y_0^+} g(y).$$

Now if either of the above inequalities were strict, then g would not be onto. Thus $\lim_{y \to y_0^-} g(y) = g(y_0) = \lim_{y \to y_0^+} g(y)$ and g is continuous at y_0. Obviously g is also right continuous at $f(a)$ and left continuous at $f(b)$, so g is continuous on $[f(a), f(b)]$.

A1016. The answer is the powers 2^k for which $2^{k+1} - 1$ is a prime. (The primes of this form are called Mersenne primes.) Note that $2^{k+1} - 1$ is always an odd integer. Thus $2^{k+1} - 1$ is composite if and only if there are positive integers a and b such that $2^{k+1} - 1 = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$. This is equivalent to $2^k = 2ab + a + b + 1 = ab + (a + 1)(b + 1)$.

What Does Random Mean?

The specifications for the Vietnam-War-era draft lotteries and for the current annual diversity immigration lottery include the word “random.” The meaning has been the subject of court cases: Is “random” supposed to refer to the process of selection or to the results, or both? Does it require equal probabilities for individuals? Is independence required?

The courts seem to be looking for a combination of fairness and equality:

- There must be “no plan, purpose or pattern in the drawing . . . perfectly fair . . . no discrimination . . . no inequity” (*U.S. v. Kotrlik, et al.*, 465 F.2d 976–977, re the 1970 draft lottery); and
- Equal probability is to be “approached as closely as reasonably possible under all the circumstances” (*Stodolsky v. Hershey*, 2 Selective Serv. L. Rptr. 3527, 3528–29 (W.D. Wis. 1969); and more recently
- The process should embody “qualities of unpredictability and equal probability” (*Smirnov, et al. v. Clinton, et al.*, United States District Court, District of Columbia, July 14, 2011, Civil Action No. 11–1126 (ABJ).)

—Paul J. Campbell

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Pitici, Mircea (ed.), *The Best Writing on Mathematics 2011*, Princeton University Press, 2012; xxx + 407 pp, $19.95 (P). ISBN 978–0–691–14841–0. *The Best Writing on Mathematics 2010*, Princeton University Press, 2011; xxviii + 380 pp, $19.95 (P). ISBN 978–0–691–15315–5.

These volumes are the first in what I hope will become an annual series; with a few exceptions, the selections were originally published in the year before the title date. The pieces included mostly look at mathematics "from the outside" and hence are non-technical; perhaps a score of pages include equations. The 2010 volume's 35 selections are organized into sections (Mathematics Alive, Mathematicians and the Practice of Mathematics, Mathematics and Its Applications, Mathematics Education, History and Philosophy of Mathematics, Mathematics in the Media), but the 2011 volume is not so subdivided. Editor Pitici provides an informative overview of each selection, and also of "runners up" that could not be included. The 2011 lead-off article is Woody Dudley's "What is mathematics [education] for?" from the *Notices AMS*, with its notably pithy answer: ". . . not for jobs. It is to teach the race to reason." (I agree with that intention but ask, in the words of a revived fast-food advertisement, "Where's the beef?"—i.e., where is the documentation that mathematics education accomplishes that aim?) Other splendid selections in the 2011 volume are Peter Denning's "The great principles of computing" and Doris Schattschneider's "The mathematical side of M.C. Escher." (The selections have been re-typeset, but figures are reproduced in black and white rather than in the original colors.)

Smirnov, et al. v. Clinton, et al.,United States District Court, District of Columbia, July 14, 2011, Civil Action No. 11-1126 (ABJ). http://dockets.justia.com/docket/district-of-columbia/dcdce/1:2011cv01126/148788/.

In its Diversity Visa program, the U.S. awards up to 50,000 permanent resident visas ("green cards") each year to individuals from countries that otherwise send relatively few immigrants to the U.S. The visas are awarded based on a lottery; for the lottery for immigration in 2012, there were 19 million(!) entries. In addition to chronic criticisms of being "riddled with fraud" and manipulated by organized crime, the program suffered further ignominy when the results were voided—two weeks after the "winners" were notified!—"due to a computer programming problem": 90% of the "winners" had applied in the first 2 days of the 30-day submission period. A new randomizer program had been used, which randomly re-ordered the entries but then still selected most of them in the order submitted. Some "winners" sued to overturn voiding the results. The law requires that the lottery be "strictly random" [as opposed to current students' loose use of "random"!]. The plaintiffs argued that "random" means, per dictionary, "without definite aim, direction, rule or method . . . haphazard," with one argument being that since it was a computer error, the results were indeed haphazard. The court held instead that Congress intended a random *process* that "embodies qualities of unpredictability and equal probability," that the lottery must be held with "similar exactitude" to state-mandated standards for slot machines(!), and that the Dept. of State was right to void the results and hold a new lottery.

Math. Mag. **84** (2011) 394–395. doi:10.4169/math.mag.84.5.394. © Mathematical Association of America

Diaconis, Persi, and Ron Graham, *Magical Mathematics: The Mathematical Ideas That Animate Great Magic Tricks*, Princeton University Press, 2012; xii + 244 pp, $29.95. ISBN 978–0–691–15164–9.

Diaconis and Graham began as entertainers (magician, juggler/trampolinist), were entranced by mathematics, and became leading mathematicians. This book shows the connections that they make between those worlds and includes how each became interested in mathematical magic (and then in mathematics). You don't need to know mathematics to get a lot out of this book, though some parts use symbolic notation and equation at some points. But the orientation toward a mathematical way of thinking is clear early on, in how the authors ask why a particular card trick works: "Just what properties of the arrangement are preserved by [the] procedure?" The authors delve into generalized de Bruijn sequences as the basis for many card tricks; explain the properties and interconnections of many kinds of shuffles; pursue an analogy between Gilbreath permutations in card shuffling and the Mandelbrot set as a "shuffle" of the plane; introduce the mathematics of juggling; and pay homage to the brilliant inventors of tricks, the "stars of mathematical magic." This engaging book encourages readers to develop improved and new magic tricks and points out open problems in the discrete mathematics involved.

Nahin, Paul, *Number-Crunching: Taming Unruly Computational Problems from Mathematical Physics to Science Fiction*, Princeton University Press, 2011; xxvi + 376 pp, $29.95. ISBN 978–0–691–14425–2.

Did you know that Alan Turing was fascinated by the Riemann hypothesis, or that Richard Feynman worked on Fermat's Last Theorem? Neither did I; these are tidbits in author Nahin's new book. He compares approaches (theoretical analysis, numerical calculation, simulation, and symbolic algebra) to simple physics problems, including temperatures on a thin plate, harmonic oscillation, the three-body problem, and electrical circuits. Shorter chapters treat a leapfrog problem, delve into the author's essays on science fiction for *Omni* magazine (what if a time traveler had gotten a calculator to Newton?), and offer a cautionary epilogue about impossible tasks. The journeys are utterly delightful, and each chapter has challenge problems for the reader (with solutions in the back); most include short programs in Matlab. The reader needs to be unafraid of differential equations, infinite series, iterated integrals, and vector algebra.

Hirsch, David, and Dan Van Haften, *Abraham Lincoln and the Structure of Reason*, Savas Beatie, 2010; xxiii + 440 pp, $34.95. ISBN 978–1–932714–89–0.

"Lincoln transformed geometry into speech." So hold this book's authors, who demonstrate how Lincoln's self-proclaimed independent study of books I–VI of Euclid formed the basis for the structure of his oratory. The authors map the six elements of a proposition (p. 29) as given in Euclid to components of a legal argument, dissect notable speeches and letters by Lincoln into those elements, and include in full text a dozen annotated speeches and letters (more than 100 pp).

Wainer, Howard, *Uneducated Guesses: Using Evidence to Uncover Misguided Education Policies*, Princeton University Press, 2011; xvi + 175 pp, $24.95. ISBN 978–0–691–14928–8.

Howard Wainer has long written the very informative Visual Revelations column in *Chance* magazine, but for even longer he worked for the Educational Testing Service and now for the National Board of Medical Examiners. In this volume he provides evidence for what happens when colleges make submitting test scores optional (namely, what you would expect to happen), for the advantages of a general aptitude test over alternatives (e.g., the SAT vs. subject achievement tests), and for the lack of validity of international comparisons; and against offering students a choice of questions to answer, against assessing teachers from student test scores, and against college rankings. The presentations are enlightening and easy to read (with some graphs, a few tables, and only two equations), and the recommendations are sensible, even as they go against conventional wisdom of educational administrators. However, readers should keep in mind that much faith in aptitude and other testing proceeds from a philosophical belief in the efficacy of a single measure of intelligence, usually called g in the literature but familiarly known as IQ.

NEWS AND LETTERS

I only just recently came across the article "Triangle Equalizers" by Dimitrios Kodokostas (April, 2010) [**2**] and realized that it shares a lot with some of the work that I have done. I thought that I might highlight some work that I have done and been exposed to regarding the problem of bisecting a triangle's area and perimeter, for others that may choose to continue in this problem area.

With respect to the problem of halving a triangle's perimeter and area, the earliest reference of this problem I have come across is that found in a 1994 article by A. Shen [**3**]. In this article Shen describes how the Soviets discriminated among prospective mathematics students by using very difficult problems. One particular example, given in 1978, was, "Draw a straight line that halves the area and perimeter of a triangle." Ilan Vardi posted a solution to this problem [**6**, solution to problem 5] and mentioned that this solution can be drawn with a ruler and compass.

Discovery of this question in Shen's article started my investigation as a senior at university which resulted in my 1999 article [**4**] which I believe compliments well the article by Kodokostas. While the articles share some of the same conclusions, my article also provides a simple means to visualize the triangles that have one, two, or three equalizers. For the curious, I might also mention that I wrote a simple interactive applet [**5**] that displays the equalizers of a triangle and relates most of the content found in my article (please forgive its simplicity as it was a side project as I was learning Java in graduate school). Also note that in reference to this Pi Mu Epsilon article, Ross Honsberger's book *Mathematical Delights* [**1**, 71–74] provides an existence proof for a triangle equalizer.

Thanks for the thread of developments about this problem cited by Kodokostas which also led me to the work of George Berzensyi and Rio Bennin—I was previously unaware of their work.

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3. A. Shen, Entrance Examinations to the Mekh-mat, *The Mathematical Intelligencer*, Vol. 16, No.4 (1994) 6–10.
4. A. Todd, Bisecting a Triangle, *Pi Mu Epsilon Journal*, Vol. 11, No. 1 (1999) 31–37.
5. A. Todd, Bisecting a Triangle, http://www.math.colostate.edu/info/applets/bisect_triangle/bisect_triangle.html
6. I. Vardi, Mekh-mat entrance examination problems, http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.html

Todd's Java applet is a delight. Among other things it displays the equalizers of an arbitrary triangle ABC with a fixed side AB, exhibiting at the same time the regions in which the vertex C has to lie in order for one or three equalizers to exist, with their common boundary being the locus of C for which two equalizers exist.

In the work of Ilan Vardi that Todd cites, there is proved the amazing fact that for any given triangle there exists an equalizer going through two specific points on its biggest and smallest sides. The construction of this equalizer can indeed be done by compass and ruler alone as Todd relates.

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Math. Mag. **84** (2011) 396. doi:10.4169/math.mag.84.5.396. © Mathematical Association of America

ACKNOWLEDGMENTS

In addition to our Associate Editors, the following referees have assisted the MAGAZINE *during the past year. We thank them for their time and care.*

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1. Publication Title: MATHEMATICS MAGAZINE	2. Publication Number: 0025-570X	3. Filing Date
4. Issue Frequency: BIMONTHLY, EXCEPT JULY + AUGUST	5. Number of Issues Published Annually: 5	6. Annual Subscription Price: $131

7. Complete Mailing Address of Known Office of Publication (Not printer) (Street, city, county, state, and ZIP+4®): MAA, 1529 18TH ST., N.W, WASHINGTON, D.C. 20036 — Contact Person: H. WALDMAN — Telephone (Include area code): 202-387-5200

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15. Extent and Nature of Circulation	Average No. Copies Each Issue During Preceding 12 Months	No. Copies of Single Issue Published Nearest to Filing Date
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i. Percent Paid (15c divided by 15f times 100)	99%	99%

16. Publication of Statement of Ownership
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